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WHAT IS WRONG WITH THE INTUITIONIST ONTOLOGY OF MATHEMATICS?

Vadim BATITSKY

The main aspect of the intuitionist ontology of mathematics is the conception of mathematical objects as products (constructions) of the human mind. This paper argues that so long as the existence of mathematical objects is made dependent on the human mind (or even any physically realizable mind), the intuitionist ontology is refutable in that it is inconsistent with our well-confirmed beliefs about what is physically possible. At the same time, it is also argued that the intuitionist's attempt to remove this inconsistency by endowing the mind with various highly idealized features and capacities will erase any significant ontological difference between Intuitionism and Platonism.

I.

As regards the ontology of mathematics, intuitionism is the view (e.g., of Brouwer in [2], [3] or Heyting in [5], [6]) that mathematical objects are products of the human mind. Where Platonists accept mathematical objects as having mind-independent non-spatio-temporal existence, and nominalists, such as Field ([4]), deny that mathematical objects exist in any sense whatsoever, intuitionists occupy a sort of an ontological 'middle ground' by maintaining that mathematical objects are mental (conceptual) entities whose existence depends (in the manner discussed below) on the human mind. In this paper, I argue that so long as the intuitionist insists that it is the human mind (or any physically realizable mind for that matter) that generates mathematical objects, the intuitionist ontology is *refutable* in the sense of being inconsistent with our accepted beliefs about the world in general and the human mind in particular. On the other hand, any attempt to avoid this inconsistency by endowing the mind with various idealized features will erase any significant difference between the intuitionist's mathematical ontology and that of the Platonist.

II.

As stated by Heyting, the central ontological, epistemological and methodological commitments of intuitionism are these:

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> ... we (intuitionists) do not attribute an existence independent of our thought, i.e., a transcendent existence, to the integers or to any other mathematical objects. ... (M)athematical objects are by their very nature dependent on human thought. Their existence is guaranteed only insofar as they can be determined by thought. They have properties only insofar as these can be discerned in them by thought. ... Faith in transcendent existence ... must be rejected as a means of mathematical proof ([6], 53).

On the ontological side, the above passage tells us that the existence of mathematical objects is dependent on human thought. What is the dependency in question? To exist, a mathematical object must be "constructible", i.e., there must be an effective procedure by which the mind can construct the said object from the antecedently constructed objects. The sequence of such constructions is grounded in our basic "intuitions" of unity, order, and indefinite repetition - the intuitions which, according to intuitionists ([5], 13-15), give rise to the sequence of natural numbers. Rational numbers, the next level of common mathematical objects, are constructed from natural numbers; real numbers are constructed from rational numbers; and so on.

The intuitionist's notion of constructibility of legitimate mathematical objects brings us to the epistemological side of intuitionism. For the intuitionist, our only legitimate epistemic access to the existence of a given mathematical object is an effective procedure for constructing that object. Accordingly, the intuitionist rejects existence theorems whose only known proofs involve inferring the existence of some mathematical object x solely on the basis of the absurdity of the supposition that x does not exist.

Finally, on the methodological side, the intuitionist's rejection of faith in the transcendent existence of mathematical objects as a means of mathematical proof is accompanied by the notion of *constructive proof* based on nonclassical (intuitionistic) logic. In intuitionistic logic, for example, the validity of the law of excluded middle is rejected. The reason is that for many a mathematical proposition we have neither an effective procedure for generating its proof nor an effective procedure for generating its refutation. Nor, as another example, does the intuitionist accept the validity of '$\neg\neg p \rightarrow p$' because we can define, say, a certain number r and a certain property P of r such that we can prove $\neg\neg P(r)$ and yet have no constructive proof of P(r) ([5], 17).

Of course, there is much more to intuitionism that is mentioned above. But beyond its philosophical underpinnings, intuitionism becomes a *mathematical research program*, of interest primarily to mathematicians who may want to investigate the "inner" mathematical issues related to intuitionistic logic or various properties of intuitionist structures in mathematics. What is important for the present discussion is whether the intuitionist's mentalistic conception of mathematical ontology is acceptable; not so much in light of its consequences for mathematical practice, but as regards the plausibility of its claims concerning the mind. The answer, as I argue below, is an emphatic 'No'.

III.

Beginning with minor things, the intuitionist tells us absolutely nothing about the ways in which the human mind might be able to construct the lowest level of mathematical reality - the sequence of natural numbers - from basic intuitions of unity, order and indefinite repetition. Nor are we told anything about what enables the human mind to have these basic intuitions. This lack of details would not in itself constitute much of a problem for the intuitionist (who is not obliged to moonlight as a psychologist) were it not accompanied by the insistence on the *irrelevance* of these details for the philosophical analysis of mathematics. For the intuitionist, it is enough to

> simply state the fact that the concepts of an abstract entity and of a sequence of such entities are clear to every normal human being, even to young children ([5], 13).

This refusal to elaborate on the fundamental philosophical assumptions about the human mind on the grounds that these assumptions "are clear to every normal human being" is most unfortunate. For unless we count Platonists as "abnormal" human beings, we have to face the sociological fact that 65% of working mathematicians are confirmed Platonists ([10]). (It is anyone's guess how many working mathematicians are *closet* Platonists.) Thus, leaving young children and the man in the street out of philosophical quibbles, the majority of "normal human beings" who do mathematics will claim that (i) *their* intuitions are quite different from those of the intuitionists and (ii) they are convinced beyond doubt, on the basis of *their* intuitions, in the transcendent (i.e., mind-independent) existence of mathematical objects. And so the intuitionist's dogmatic claim about basic intuitions of the mind is in very deep trouble as a matter of sociological fact about mathematical practice.

But there is something far more troubling about intuitionism. Suppose that the intuitionist (or his friends in the psychology department) manage to produce some convincing account of how the basic intuitions of unity, order and indefinite repetition come to be possessed by the human mind; and, also, how the mind constructs natural numbers from the intuitions in question. What I wish to maintain is that so long as this account of the human mind is consistent with the rest of our knowledge about the world, the human mind, or even the mind of a *physically ideal cognitive agent*, will be able to construct at most an infinitesimally small part of the *intuitionistically legitimate* mathematical reality because it will be able to generate only an infinitesimally small part of the *intuitionistically acceptable* proofs.

To see this, let us consider, as a very simple test case, a mathematical proposition which the intuitionist accepts as provable (and, hence, true). Let 'Pr(x)' be 'x is prime' and consider

(*) $\mathrm{Pr}\,(2^{1297} - 11) \vee \neg\, \mathrm{Pr}(2^{1297} - 11)$.

Because 'Pr' is a decidable property of natural numbers, the disjunction (*) is accepted by the intuitionist as provable because the decidability of 'Pr' guarantees either a constructive proof of the left disjunct of (*) or a constructive proof of the right disjunct of (*). The problem, however, is that executing the simplest deterministic algorithm for primality when the number involved is as large as that in formula (*) requires so many steps that a *physically ideal* computer, (i.e., the size of the universe densely packed with atom-sized switching gates operating at the speed of light, etc.) would have to spend more than the estimated entire lifetime of the universe to produce the answer. (See [9], [11] and [12] for the complexity-theoretic obstacles to implementations of such algorithms.) Needless to say, the human mind (even the legendarily quick mind of von Neumann) will fare much worse in effecting the required construction. Yet (*) *is* provable for the intuitionist, which means that it is time to start worrying about the kind of a "mind" that the intuitionist speaks of as the generator of constructive proofs.

The intuitionist's usual reply to such worries is that so long as an effective procedure can be carried out *in principle*, i.e., by an *ideal agent* who does not suffer from the biological limitations to which actual human agents are subject, the proof generated by this procedure is perfectly acceptable.

This, for example, is the view of Kitcher who tells us that mathematics *owes its truth not to the operations of actual human agents, but the ideal operations performed by ideal agents* ([8], 109). But what is the idealization involved here? After all, we have powerful and well-confirmed scientific theories telling us that, with the speed of light and the spatio-temporal extent of the universe being finite, applications of effective procedures to all but an infinitesimally small fraction of instances of non-trivially solvable decision problems will remain beyond the reach of anything that obeys the constraints of known physical laws (let alone the laws of human biology). Obviously, then, the idealization in question has to free the cognitive agent from the limitations to which everything in the universe is subject. Put differently, while the Platonist remains prudently non-committal about the specific means by which the human mind accesses mathematical objects (assuming, as Gödel does in [7], only some sort of "mathematical sense"), the intuitionist's mentalistic foundation for mathematical ontology forces him to attribute to the mind capacities which are not only patently nonhuman in their prodigiousness, but also patently unrealizable by any physically possible (ideal) structure whatsoever.

Again, this does not seem to pose a difficulty for some philosophers sympathetic to intuitionism who propose that the ideal agent's performance of computational tasks of arbitrary complexity is best viewed as *carried out in a medium* analogous *to time, but far richer than time* ([8], 146).

What then is this 'mind' which underlies the intuitionist's ontology of mathematics? Certainly not the human mind as we know it; nor it is the mind of any *physically ideal* creature which may inhabit the universe. (Otherwise, the intuitionist's ontology rests on patently false assumptions.) Instead, it turns out to be the mind whose computational capacities are *mathematically ideal* (i.e., they encompass computational tasks of arbitrary complexity) and are exercised in a *mathematical universe* (which, unlike the known physical universe, can accomodate arbitrary long computations either by allowing arbitrarily fast operations or by offering enough time for completion of computational tasks of arbitrary length).

But in this case, the intuitionist's conception of the mind as the source of mathematical objects, if coherent at all, survives only by replacing the Platonist's transcendent existence of mathematical objects with the transcendent existence of what amounts to nothing less than 'the mind of God' or, less theologically, the Universal Turing Machine. For it is easy to see that the Universal Turing Machine has precisely the idealized capacities that the intuitionist requires from the mind: infinite supply of tape for carrying out arbitrarily long computations, freedom from the physical constraints of space and time, freedom from functioning errors due to deterioration of hardware, etc. As a result, there is no way for the intuitionist to accept the existence of such a 'mind' without shooting himself in the foot, so to speak, if only because, *from the ontological point of view*, there is no interesting difference between accepting the existence of the Universal Turing Machine (or some equally abstract entity) as the generator of mathematical objects and accepting mathematical objects as happily existing on their own in Plato's Heaven.

Department of Philosophy
St. Johns University, New York
300 Howard Avenue
Staten Island, NY 10301, U.S.A.
Fax: 718-390 4347, E-mail: batitsky@simcl.stjohns.edu

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