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# Tichý's Two-Dimensional Conception of Inference

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**Abstract:** In this paper we revisit Pavel Tichý's novel distinction between one-dimensional and two-dimensional conception of inference, which he presented in his book *Foundations of Frege's Logic* (1988), and later in *On Inference* (1999), which was prepared from his manuscript by his co-author Jindra Tichý. We shall focus our inquiry not only on the motivation behind the introduction of this non-classical concept of inference, but also on further inspection of selected Tichý's arguments, which we see as the most compelling or simply most effective in providing support for his two-dimensional account of inference. Main attention will be given to exposing the failure of one-dimensional theory of inference in its explanation of indirect (reductio ad absurdum) proofs. Lastly, we discuss shortly the link between two-dimensional inference and deduction apparatus of Tichý's Transparent Intensional Logic.

**Keywords:** Deduction – Frege – Gentzen – indirect proofs – Tichý – TIL – two-dimensional inference.

## 1. Introduction

In his *Foundations of Frege's Logic* (1988) Pavel Tichý offered quite unusual conception of inference (deduction), which he dubbed as two-dimensional inference. Our main purpose here will be to provide further examination of this atypical notion of inference that stands in sharp contrast to "traditional" one-dimensional one. This will go hand in hand with our reexamination of selected Tichý's arguments, which we see either as the most potent or most convincing for the case of accepting the two-dimensional account of inference.

The following paper is structured into three parts: the first one is devoted to the general introduction of two-dimensional inference and its main properties. The second, and main, part will be dealing with arguments Tichý presented in order to vindicate his new-found dichotomy, as well as with the rationale behind it. In the third, and final, part will be briefly discussed the relationship between two-dimensional inference and Tichý's Transparent Intensional Logic (TIL).

## 2. Two-dimensional inference: a brief overview

Before we approach the two-dimensional inference itself it will be very useful to first shortly recount Tichý's general conception of inference. In Tichý (1988, 235-236) we can find three basic characteristics, which he uses to describe inference: inference is (a) advancement from some premises to what they entail, (b) a way of extending our knowledge and (c) a truth-proliferating operation. So for Tichý inference is not only an operation (or a function; see Tichý – Tichý 1999, 73) that preserves truth, but also extends our knowledge base. It is also important to note that when Tichý talks about truth proliferation, he means proliferation of *logical* truths in broad sense (valid entailments, tautologies, theorems), rather than the empirical ones.[1] Remember these features of inference well as they will come in handy later.

Now, once we have briefly familiarized ourselves with Tichý's general take on inference, we can focus our attention specifically to the two-dimensional case. As already hinted, Tichý distinguishes between the so-called "one-dimensional" and "two-dimensional" view on inference, or more precisely, on the role which hypotheses play in deduction (see Tichý 1988, 235).

Let's begin with the one-dimensional account of inference:

1 This might also explain why Tichý prefers the term "truth-proliferation" to the much more common "truth-preservation": in his account we really are rather expanding the inference than just keeping it valid, simply because every step is true from the very beginning, so it only makes sense to speak of proliferation instead of preservation. Of course, this distinction is merely stylistic. See also Tichý – Tichý (1999, 74).

> On one view, inference steps take hypotheses themselves as premises and yield what those hypotheses entail. This might be called the *one-dimensional* view of inference. (Tichý 1988, 235)

On the two-dimensional account, however, the inference steps do not proceed from hypotheses to conclusion, i.e., from some proposition or propositions to another one. That's because the building blocks of two-dimensional inference are not propositions.

> On the other, *two-dimensional*, view inference steps do not work on hypotheses as such but on antecedents consequent compounds, i.e., on entailments in which the hypotheses appear as antecedents. As inference step takes us then from one or more valid entailment of this sort to a further valid entailment. (Tichý 1988, 235)

So on the *one-dimensional* account the inference step takes us from certain proposition(s) to another proposition, while on the *two-dimensional* account the inference step takes us from certain valid entailment(s) to another valid entailment. In other words, for Tichý inference is an operation on valid arguments (antecedents/consequent compounds), not on their constituents, i.e., antecedents and consequents. The difference between one-dimensional and two-dimensional inference can be graphically illustrated in the following manner:

One-Dimensional Inference:

$$\frac{\text{hypothesis}}{\text{conclusion}}\ \textit{Inference Step}$$

Two-Dimensional Inference:

$$\cfrac{\cfrac{\text{antecedents}}{\text{consequent}}}{\cfrac{\text{antecedents}}{\text{consequent}}}\ \textit{Inference Step}$$

As we can see, what Tichý is actually doing is combining "hilbertian" style of proving from logical truths, i.e., axioms, with "gentzenian" style of proving from assumptions. So in the end we get a deduction method, where we start proving from logically true assumptions, i.e., antecedents/consequent compounds (valid entailments), and continue to other valid entailments, which logically follow from them.[2]

2 It seems that what is crucial in Tichý's theory of inference is not really the two-dimensionality of inference steps, but rather their "self-sustained" nature, i.e., their

It is worth to note that Tichý's terminology is here very fluctuating: aside from antecedents/consequent compounds and valid entailments, he also speaks of conditionals, tautologies and theorems. Granted, they all can be viewed—at least from Tichý's standpoint—as referring to one and the same thing. For simplicity sake, we will prefer the first two terms, i.e., antecedents/consequent compounds and valid entailments (or arguments) and ignore the rest.

To sum it up, the two-dimensionality of inference lies in the idea that we do not infer from various hypotheses to their logical conclusion, but from valid arguments, composed of hypotheses and their conclusions (i.e., antecedents and consequents), to other valid argument. In other words, we move one dimension up, hence two-dimensional conception of inference, in which the inference "atoms" are no longer propositions, but the whole valid arguments built from them.

Finally, let's consider the following argument example:

Premise 1: It rains.
Premise 2: If it rains, the streets will be wet.
———————————————————
Conclusion: The streets are wet.

What we learn from this and other similar valid arguments, according to Tichý, is not that "The streets are wet", but the whole (logical) fact that "From 'It rains' and 'If it rains, the streets will be wet' follows that 'The streets are wet.'", which he calls *entailment statement* (Tichý 1988, 236). In other words, we learn no empirical fact by simply carrying out the inference, but *only* that between such and such propositions holds relation of logical consequence.[3]

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autonomous logical validity. This, of course, raises a couple of further questions well-fitted for further study, e.g., why should be the premises of deduction always logical truths or what are we exactly doing when we move from antecedents to consequent (it cannot be inference, since it is reserved for moves on "higher" dimension).

[3] This, however, doesn't mean that we are unable to learn any new empirical truths through the two-dimensional inference at all: although it doesn't really make sense to say "if the premises are true, then the conclusion is true", because premises are always valid entailments (tautologies), it still makes sense to say "if the empirical propositions that appear in the premises are true, then the empirical proposition that appears in the conclusion is true". Remember that on the two-dimensional account of inference, premise is the whole antecedents/consequent compound. See also Tichý – Tichý (1999, 73-75).

The new piece of information that expands our knowledge base is then not the conclusions itself, but the logical truth that such conclusion follows from such premises (i.e., the entailment statement).

Or, to use Tichý's own example, let's have the following proposition:

(1) Peter and Paul are spies.

From this, Tichý argues, we cannot infer

(2) Peter is a spy.

but only the whole entailment statement:

From "Peter and Paul are spies" follows that "Peter is a spy".

The reasoning is the following: let's assume that the inference step in question would really take us from (1) to (2). What would we learn by such a move? That Peter is really a spy? Certainly not, because (1) might be a purely hypothetical statement. Thus, one-dimensional explanation of this argument fails to meet Tichý's second requirement for inference (b), i.e., that it expands our knowledge. In other words, by inferring (2) from (1) we learn nothing at all about Peter being a spy: but what we learn, is that (2) follows from (1).

Now remember the third condition (c): inference must be truth-proliferating. It's easy to see that the one-dimensional account fails to satisfy even this requirement. For something to be truth-proliferating it must be applied to something that is true (otherwise what should it proliferate?). But in this case, we have no knowledge whatsoever whether (1) is true or not. In other words, (1) is simply a hypothetical assumption and as such it needs no concrete truth value. Thus, the move from (1) to (2) can't be inference, since it does not proliferate truth. So it seems that the only condition that the one-dimensional view of inference can fulfill is the first one (a). And one out of three, that's hardly a satisfactory result.

To summarize the first section, let us say the following: Tichý's inference proceeds not from hypotheses to conclusion, but from valid argument(s) to other valid argument. This is the core of two-dimensionality, i.e., that the corner stones of deduction are valid entailments.

In the next section we examine more closely the arguments in support of two-dimensional inference and try to shed further light on the whole motivation behind it.

## 3. Motivation in the background

Why Tichý strives for the vindication of this atypical, novel conception of inference? What are its main advantages? What bothered him so much about the classical one-dimensional inference (aside from the already discussed matter that it fails to satisfy two out of three of his own general requirements for inference)? These are the questions we try to answer in this section.

Tichý's reasons for introducing two-dimensional inference can be broadly categorized in two groups: (i) logical (formal, technical) ones and (ii) epistemological (philosophical) ones. The logical reasons contain, e.g., unsatisfactory (at least for Tichý) explication of indirect proofs offered by the one-dimensional account of inference. Among the epistemological reasons we can include, e.g., the impossibility of inference from false propositions (i.e., continuation in Frege's line of thought; see, e.g., Frege 1914, 244-245), problems affiliated with the introduction of assumption as cognitive attitude *sui generis* (see Tichý 1988, 254) or complications accompanying analysis of natural language arguments involving fictional characters or arbitrary objects.[4]

Given that the main subject matter of this paper is deduction, we will focus here only on the first mentioned group, more accurately, on the failure of one-dimensional account of indirect proofs, which will be the topic of our inquiry below. Although, it's important to keep in mind that this distinction serves mainly a didactic purpose and in reality both groups (i) and (ii) are, of course, very closely related and intertwined.

One last thing that needs to be said before we move forward to the examination of the just mentioned failure is that we will not be echoing Tichý's original argumentation step by step. Rather, we present here only some of his arguments. More specifically, those which we see as the most persuasive, comprehensive and easily digestible and we try to expand on them further a little.

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[4] Both topics are discussed at length in *Chapter 14: The Fallacy of Subject Matter* in Tichý (1988).

### *3.1. Failure of one-dimensional inference*

As we have already implied above, according to Tichý the one-dimensional theory of inference is incapable of precisely describing indirect (reductio ad absurdum) proofs. Let's check if it is really the case.

Suppose that we want to offer an indirect proof of the following mathematical statement:

If $x = 3$ then $2x + 4 \neq 12$

First step is to assume that its opposite holds, i.e.,

If $x = 3$ then $2x + 4 = 12$

If we proceed to solve the equation $2x + 4 = 12$, we learn that $x = 4$. Now, if we put it back to the original statement, we get that

If $x = 3$ then $x = 4$

which is, of course, a contradiction. Thus, our reductio assumption that if $x = 3$ then $2x + 4 = 12$ must be false and its negation true. Therefore, we have proved that if $x = 3$ then $2x + 4 \neq 12$. ∎

But take note of the fact that from what we infer in the end that if $x = 3$ then $2x + 4 \neq 12$ is not just series of individual steps, but rather the whole preceding argument, i.e., the argument that has just resulted in contradiction. What we do in the last step of reductio proof is that we withdraw of one of the premises (the one that has led us into contradiction) and then we put its negation as a conclusion of another argument, i.e., the argument which does not have among its antecedents this particular reductio hypothesis. From this we can see that we are not really dealing here with inference between propositions alone, but rather with inference between two arguments.

Or to put it differently, reductio proof is guided by the following rule: "Put the opposite of anticipated conclusion among antecedents, and if you end up in contradiction, infer another argument, which has as its consequent the anticipated conclusion." From the wording of this instruction it is apparent that we are really moving from one argument (the failed one, i.e., the one with contradiction) to another, not just from proposition(s) to other proposition. In other words, from the failure of one argument we infer another one, in which the negation of reductio hypothesis appears as a consequent.[5] However, this type of inferring one argument from another is something which cannot be fully explained in terms of the one-dimensional view, where we just work with propositions.

Of course, the one-dimensional theory of inference can describe indirect proofs, but Tichý's point is—and I think he is quite right in this—that its description is inaccurate and doesn't really correspond with what we are actually doing while we are carrying out indirect proofs such as the one just mentioned. In this respect, the one-dimensional account of indirect proofs seems inadequate.

So what form should take the adequate rule for reductio proofs? Before we try to answer this, we will make a short detour to proofs in general.

As we have already repeated above many times, according to Tichý we don't prove things from hypotheses, but from compounds assembled from hypotheses (antecedents) and single conclusion (consequent). This also means, among other things, that during the proving process each inference step in a way recapitulates all the hypotheses of previous steps (i.e., which hypotheses are still in force, and which were abandoned).

Therefore, proof is better seen as composed of consecutive stages, i.e., gradually expanding valid arguments, each of which is fully self-contained (see Tichý – Tichý 1999, 75), rather than as just single statements. This is very noticeable in Fitch diagram proofs. Let's have a look at the following example that uses Tichý as well:

| | | |
|---|---|---|
| 1 | $[p \supset s]$ | hyp |
| 2 | $[p \supset s] \supset [p \supset [p \supset r]]$ | hyp |
| 3 | $[p \supset [p \supset r]]$ | 1, 2, mp |
| 4 | $\quad p$ | hyp |
| 5 | $\quad [p \supset [p \supset r]]$ | 3, reiteration |
| 6 | $\quad [p \supset r]$ | 4, 5, mp |
| 7 | $\quad r$ | 4, 6, mp |
| 8 | $[p \supset r]$ | 4-7, implication introduction |

5 In indirect proofs we take into account the whole argument, not just its conclusion, because the conclusion alone would not be able to justify why should hold the opposite of the reductio hypothesis.

Notice that even though the last 8th line is justified only with reference to subproof on lines 4 to 7 (plus the accompanying inference rule), if we would really like to check its correctness, we would have to take into account also its "position" in the whole proof. More precisely, we would also have to look onto the line 3, to check if the line 5 is correct. Tichý writes:

> The point is that the notion of subordinate proof is not absolute but relative to the particular place that a subproof occupies in the main proof. What is a subproof as it occurs in a particular proof in a particular place, may not be a subproof as it occurs in another proof or in a different place in the same proof. (Tichý 1988, 246)

This leads Tichý to reinterpreting proofs as, rather than moving from individual propositions to another propositions, as progressing from one segment (i.e., antecedents/consequent compound) of the proof to another segment. Tichý then continues:

> Individual constituents of a proof must not be construed as single statements ("propositions" in Fitch's terminology) or subproofs, but as antecedents/consequent compounds. (Tichý 1988, 249)

This is also the reason why Tichý chooses to base his deduction apparatus on Gentzen's sequent calculus (see Gentzen 1934): it (at least according to Tichý) explicitly embodies and captures his idea of two-dimensional inference.

In contrast to Gentzen, however, Tichý sees sequents not as just two strings of "unconnected" formulae, i.e., antecedents on the left side and succedents on the right side, but (and this will hardly come as a surprise) as valid entailments, i.e., antecedents/consequent compounds. We will denote Tichý's sequents in the following way

$$\alpha_1, \ldots, \alpha_n / \chi$$

where $\alpha_1, \ldots, \alpha_n$ are antecedents and $\chi$ consequent.

Now, if we apply this sequent style of proofs to our earlier example in Fitch notation, we will see that the last line 8 is no longer warranted, because not all relevant hypotheses have been listed. The last step of the proof then should look more like this

$$[p \supset s], [p \supset s] \supset [p \supset [p \supset r]], p / [p \supset r]$$

i.e., it should contain all the hypotheses, which are still in force.

In other words, according to Tichý, for full and comprehensive description of a proof step it is not enough just to list the conclusion and the immediate lines, from which it was inferred, but also all the hypotheses that are still assumed. Tichý writes:

> [A] step in a proof is not completely described by simply citing the succedent formula B. The nature and legitimacy of a step depends equally on what particular hypotheses are currently in force. Besides, a step often consists in discharging a hypothesis and leaving the succedent intact. Thus in a fully perspicuous proof, where nothing is suppressed, the relevant hypotheses have to be listed at each step. (Tichý 1988, 251)

And according to him ignoring this leads to the following mistake:

> This leaves the door open for a kind of double talk. It makes it possible to imagine that in a proof formalized as a string of sequents it is the succedent of a step that is inferred from the succedents of some preceding steps, rather that the whole sequent from the preceding sequents. An illusion is thus created that the premises of an inference are often purely hypothetical statements which the maker of the inference would not dream of endorsing of subscribing to. (Tichý 1988, 253)

Now we can finally return back to our unfinished business from earlier and try to formulate basic scheme for reductio proof in scope of two-dimensional inference. The rule for indirect proof can be in simplified manner stated in the following way:

$$\frac{\alpha_1, \ldots, \alpha_n, \neg\rho \;/\; \bot}{\alpha_1, \ldots, \alpha_n \;/\; \rho}\ \textit{Inference Step}$$

where $\bot$ is contradiction and $\neg\rho$ reductio hypothesis.

This brings us to the end of second part. The last topic that remains to be discussed is the relationship between two-dimensional inference and TIL.

## 4 Two-Dimensional Inference and TIL

At first sight it seems that two-dimensional inference is rather standalone concept, independent not only of TIL, but also hyperintesionality in general. Is it really so, or does two-dimensional inference actually offer anything TIL specific?

Here the situation gets complicated, because Tichý himself never simultaneously discussed both two-dimensional inference and the deduction system of TIL itself, which is based around the concept of *match* (see, e.g., Tichý 1982b). Of course, when discussing the match and the rest of the deduction system, Tichý also relies on generalized version of sequent calculus (after all, he calls the antecedents/succedent entity as sequent), but on the other hand, the adoption of sequent calculus doesn't necessarily mean also the acceptance of the two-dimensional inference, as was evident in Gentzen's work.

To put it differently, Tichý's deduction apparatus that appeared in his earlier works predating Tichý (1988), i.e., Tichý (1982), (1982b), (1986), can be quite easily interpreted even in terms of one-dimensional account of inference. In this respect, it would seem that the introduction of two-dimensional inference was motivated mainly by Tichý's pursue for overall philosophical rigor rather than by something strictly TIL related.

This interpretation would be also supported by the fact that Tichý repeatedly talks about two-dimensional and one-dimensional *view* on inference, not two different kinds of inference. If we take this Tichý's formulation seriously, it will become clear, that the two-dimensional and one-dimensional accounts of inference are not so much two competing concepts, but rather two distinct tools for two distinct scales. For some straightforward deductions, the one-dimensional approach might be (and actually is) sufficient, but for some other, more complex ones, it seems to fail to offer satisfactory explanation, as we have seen in a case of indirect proofs.

Simply put, adhering to one position does not necessarily compromise the other one. It is rather a question of accuracy and scrupulousness of explication of inference. In this respect, we can simply view the two-dimensional inference just as more fine-grained, more precise analysis of the "traditional" one-dimensional inference that Tichý developed in order to adequately describe reductio proofs. But as already stated, for some tasks the latter might work just fine (and sometimes even better), just as for some tasks there is no need for microscope, because magnifying glass will suffice.

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