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An Outline of a Substructural Model of BTA Belief[1]

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Abstract: The paper outlines an epistemic logic based on the proof theory of substructural logics. The logic is a formal model of belief that *i*) is based on true assumptions (BTA belief) and *ii*) does not suffer from the usual omniscience properties.

Keywords: Belief – epistemic logic – logical omniscience – substructural logics.

1. Introduction

The Gettier examples (see Gettier 1963) suggest that in order to know a proposition, the belief that the proposition holds cannot be based on false assumptions. For example, assume that my colleague Dr. A has bought a new car. I believe that one of my colleagues has bought a new car (*C*), but this belief is based on the false assumption that Dr. B, also a colleague of

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mine, is the owner (B). My belief that C is a true belief. It is also justified: by valid inference from B and $B \rightarrow C$. However, it is quite plausible to assume that I *do not know* that C. The obvious reason is that my justification of the belief that C is based upon the false assumption B.

The notion of *belief based on true assumptions* (BTA belief) is interesting even in itself, considered independently of the analysis of *knowledge* (and even independently of the Gettier examples). BTA belief is *safe*: if a justification is requested, true assumptions can be provided. Moreover, if "based on" is construed as a truth-preserving relation, then BTA belief yields *true belief*.[2]

This paper outlines a simple formal model of BTA belief. The basic idea is to use *explicit* bodies of assumptions (or information) and the consequences of portions of this body. Hence, "based on" is construed as a truth-preserving relation. However, it is not assumed that the body of assumptions is a set and that the consequence relation is classical: the model utilises the proof theory of *substructural logics*.[3]

Section 2 briefly reviews the possibilities of modelling BTA belief within standard epistemic logics. Section 3 outlines the substructural approach: epistemic states are defined and the semantics of a propositional epistemic language is given. Section 4 provides examples of valid formulas and discusses some prominent examples of non-valid formulas. Section 5 concludes the paper and outlines directions of future work.

2. Epistemic logics and BTA belief

Standard epistemic logics (see Fagin et al. 1995, ch. 1-3, for example) model belief as a necessity-like operator. Semantically, the logics correspond to various classes of models $M = (W, R, V)$, where W is a non-empty set, R is a binary relation on W and V is a valuation. The truth conditions of Boolean formulas in points $x \in W$ are the usual Boolean conditions. A formula A is *believed* in x iff A holds in every y such that Rxy.

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2 On the other hand, if "based on" is construed, for example, in probabilistic terms, then this does not hold: a set of true propositions can make a proposition p "highly probable" while p is, in fact, false.

3 An exposition of substructural logics can be found in Restall (2000).

One may provide a BTA-like interpretation of this semantics. Assume that every x is given a body of formulas $At(x)$, seen as the *assumptions* adopted in x. The relation R may be construed as follows: Rxy iff every formula in $At(x)$ holds at y. If R is assumed to be *reflexive*, then every assumption adopted in x is true in x.

However, there are two problems. First, the bodies of assumptions $At(x)$ are not explicitly given within standard epistemic models. In general, standard epistemic logics do not articulate *reasons* for beliefs. Second, the resultant belief operator suffers from the notorious omniscience properties.[4] For example, belief is closed under every propositionally valid inference rule. As a special case, every propositional tautology is believed. In general, belief within a standard epistemic logic L is closed under every L-valid inference rule.

Both these problems are addressed by *justification logics* (see Artemov 1994, 2001, 2008 and 2011, for example). These extend the Boolean language by a set of *justification terms*. This allows to express claims such as "t justifies A", where t is a justification term and A is a formula. Informally, "t justifies A" is true in a world x only if t is an admissible evidence for A at x (or: relatively to the context of x). It has been proposed recently to interpret justification terms t as sets of formulas $*(t, x)$, relatively to worlds x (Artemov 2012). In line with this interpretation, t may be seen as corresponding to a true assumption at x iff every formula in $*(t, x)$ is true in x.

3. Epistemic states and substructural logics

This section outlines the basics of the substructural approach. Epistemic states are defined and an interpretation of the formal definition is briefly discussed. Familiarity with substructural logics is helpful, but the necessary background is provided.

Definition 3.1

The *language* L_0 is the language of classical propositional logic. The *language* L_1 is L_0 with a unary operator **Bel**. p, q, etc. (A, B, etc.) will be used as metavariables ranging over the set of propositional variables P (the set of formulas Fm). Every formula of L_0 is a *structure* and the only

[4] A readable discussion of the omniscience properties may be found in Fagin et al. (1995, ch. 9).

substructure of itself. If X and Y are structures, then (X ; Y) is a structure with substructures X and Y (We shall refrain from using the outermost pair of parentheses). The set of structures will be referred to as *Struct*. A *consecution* is an expression of the form $X \vdash A$, where X is a structure and A is a formula of L_0. A *structural rule* is a rule of the form

$$\frac{Y(X) \vdash A}{Y(X^{*}) \vdash A}$$

It is assumed that structural rules are closed under substitution of formulas. (Structural rules shall be referred to by $X \Leftarrow X^{*}$.) If Γ is a set of structural rules (closed under derivability[5]), then a consecution is Γ-*provable* iff it is provable using no other structural rules than those in Γ.

The intuitive interpretation of **Bel** is "it is a BTA belief that". Structures are to be seen as bodies of information, where ";" is a punctuation mark meaning "taken together with". Consecutions $X \vdash A$ are read "the structure X entails A". Structural rules state that certain structures X may be replaced (even within other structures Y) by structures X^{*} without affecting the set of entailed formulas. Here are some familiar examples of structural rules:

(B)	$X ; (Y ; Z) \Leftarrow (X ; Y); Z$	("Associativity")
(Bc)	$(X ; Y); Z \Leftarrow X ; (Y ; Z)$	("Converse associativity")
(CI)	$X ; Y \Leftarrow Y ; X$	("Weak commutativity")
(M)	$X \Leftarrow X ; X$	("Mingle")
(WI)	$X ; X \Leftarrow X$	("Weak contraction")
(K)	$X \Leftarrow X ; Y$	("Weakening")

We shall be working with *natural deduction* systems for substructural logics. These systems are given by the axiom $A \vdash A$, a set of structural rules and a set of introduction and elimination rules for the connectives. While variations in the former yield different substructural logics, the latter will be constant:

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5 A rule R_0 is derivable from rules R_1, ..., R_n iff the admissibility of R_0 is a consequence of the assumption that R_1, ..., R_n are admissible. For example, (M) is derivable from (K): if X ; Y may replace X, where Y is arbitrary, then, obviously, X ; X may replace X as well.

(→I) If $X ; A \vdash B$, then $X \vdash A \to B$.
(→E) If $X \vdash A \to B$ and $Y \vdash A$, then $X ; Y \vdash B$
(∧I) If $X \vdash A$ and $X \vdash B$, then $X \vdash A \wedge B$.
(∧E) If $X \vdash A \wedge B$, then $X \vdash A$ and $X \vdash B$.
(∨I) Both $X \vdash A$ and $X \vdash B$ yield $X \vdash A \vee B$.
(∨E) If $Y(A) \vdash C$, $Y(B) \vdash C$ and $X \vdash A \vee B$, then $Y(X) \vdash C$
(Neg) If $A \vdash \neg B$ and $X \vdash B$, then $X \vdash \neg A$

Let us note that the cut rule

(Cut) If $X \vdash A$ and $Y(A) \vdash B$, then $Y(X) \vdash B$

is admissible in every natural deduction system built upon these rules.

Of course, structural rules affect the set of provable consecutions. For example, weakening is essential in the proof of $A \vdash B \to A$:

1. $A \vdash A$ (axiom)
2. $A ; B \vdash A$ (1., K)
3. $A \vdash B \to A$ (2. →I)

Similarly, (CI) is sufficient for $A \vdash (A \to B) \to B$ and (B), (Bc), (WI) are sufficient for $A \to (A \to B) \vdash A \to B$. The choice of structural rules is usually influenced by the *assumed nature* of structures. For example, if structures are seen as *sets*, then all the above structural rules (with the possible exception of (K)) are plausible.[6] If it is assumed in addition that the consequence relation is monotonic, (K) is plausible as well. However, if structures are seen as *multisets*, then (M) and (WI) are no longer plausible. If they are seen as *lists*, then (CI) has to go as well. For more detail see Restall (2000).

Definition 3.2
A substructural *frame* is a tuple $F = (W, R, C, \leq)$ where W is a non-empty set, R is a ternary relation on W, C is a symmetric[7] binary rela-

6 The set $\{A, B\}$ is identical with $\{B, A\}$ and $\{A, A, B\}$.

7 Symmetry is assumed since we shall be working with a single negation. In addition, we opt to consider a single implication, but without considering only commutative frames. This is the case since we want our "non-epistemic fragment" of the language to consist of the ordinary Boolean formulas. Symmetry is a natural assumption if C is read

tion on W and $\leq$ is a partial order on W. Moreover, the following conditions are assumed:

If $Rxyz$ and $x' \leq x$, $y' \leq y$, $z \leq z'$, then $Rx'y'z'$.
If Cxy, $x' \leq x$ and $y' \leq y$, then $Cx'y'$.

A *substructural model* is a couple $M = (F, E)$, where E is an *evaluation function* from $W \times (Fm \cup Struct)$ to $\{0, 1\}$ such that:

If $E(x, p) = 1$ and $x \leq y$, then $E(y, p) = 1$
$E(x, A \wedge B) = 1$ iff $E(x, A) = E(x, B) = 1$
$E(x, A \vee B) = 1$ iff $E(x, A) = 1$ or $E(x, B) = 1$
$E(x, A \rightarrow B) = 1$ iff for all y, z: If $Rxyz$ and $E(y, A) = 1$, then $E(z, B) = 1$
$E(y, \neg A) = 1$ iff for all y: Cxy implies $E(y, A) = 0$
$E(y, X ; Y) = 1$ iff there are y,z: $Ryzx$, $E(y, X) = 1$ and $E(z, Y) = 1$

A consecution $X \vdash A$ is valid in M iff $E(x, X) = 1$ implies $E(x, A) = 1$ for all $x \in W$. A consecution $X \vdash A$ is valid in F iff it is valid in every $M = (F, E)$.

We do not give an exposition of the substructural semantics – interested reader is referred to Restall (2000) and Mares (2004).

Lemma 3.3
Let ND be a substructural natural deduction system with a set Γ of structural rules. A consecution $X \vdash A$ is provable in ND iff it is valid in the class of frames that satisfy the conditions corresponding to members of Γ. Some of the corresponding conditions are:

c(B) *If $R(xy)zw$, then $Rx(yz)w$*
c(Bc) *If $Rx(yz)w$, then $R(xy)zw$*
c(CI) *If $Rxyz$, then $Ryxz$*
c(M) *If $Rxxy$, then $x \leq y$*
c(WI) *$Rxxx$*
c(K) *If $Rxyz$, then $x \leq z$*

(*$R(xy)zw$ means that there is a u such that $Rxyu$ and $Ruzw$, while $Rx(yz)w$ means that there is a u such that $Ryzu$ and $Rxuw$.*)

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as "consistency", but commutativity is considered to be rather strong. For details, see Restall (2000).

Proof: See Restall (2000, ch. 11).

Definition 3.4
Let Γ be a set of structural rules. A *Γ-frame* is a frame that satisfies every condition corresponding to members of Γ. A *Γ-countermodel* to $X \vdash A$ is a model $M = (F, E)$, where F is a Γ-frame and there is $x \in W$ such that $E(x, X) = 1$ but $E(x, A) = 0$.

Definition 3.5
An *epistemic state s* is a triple (X, Γ, V), where X is a structure, Γ is a set of structural rules and V is a function from P to $\{0,1\}$. *Truth values* of L_1-formulas at states are defined as follows:

$T(s, p) = 1$ iff $V(s, p) = 1$
$T(s, \neg A) = 1$ iff $T(s, A) = 0$
$T(s, A \wedge B) = 1$ iff $T(s, A) = 1$ and $T(s, B) = 1$
$T(s, A \vee B) = 1$ iff $T(s, A) = 1$ or $T(s, B) = 1$
$T(s, A \rightarrow B) = 1$ iff $T(s, A) = 0$ or $T(s, B) = 1$
$T(s, \mathbf{Bel}\, A) = 1$ iff there is a substructure Y of X such that $Y \vdash A$ is Γ-provable and $T(s, B) = 1$ for every formula B in Y.

A formula is *Γ-valid* iff it is true in every $s' = (X', \Gamma, V')$. A formula is *universally* valid iff it is Γ-valid for every Γ.

An epistemic state is given by a structured body of *information* (assumptions) X and an "*environment*" specified by the valuation V. Notice that we construe epistemic states as *syntactic* objects. This is an alternative to the usual construal of states as "sets of possible worlds".

As noted above, the properties of X are partially specified by the structural rules in Γ. It is a BTA belief that A iff there is a body of information Y within X such that *i)* A is inferable from Y using no structural rules other than those in Γ and *ii)* Y consists only of *true* formulas.[8]

Let us note that this approach builds upon Konolige's deductive model of belief, see Konolige (1984), and the usual methods of knowledge representation, see Šefránek (2000), Brachman – Levesque (2004). However, the present framework is a generalisation of these: substruc-

[8] The truth condition of **Bel** is the reason why structures ("assumptions") may contain only L_0–formulas. A more generous definition of structures would render the truth condition circular.

tural logics allow us to work with various types of structures and consequence relations. The idea of using substructural logics in modelling epistemic notions is not new, see Bílková et al. (2010) and Sequoiah-Grayson (2009), for example.

4. Properties of BTA belief

This section discusses some of the properties of BTA belief. We begin by pointing out several *universally* valid formulas and rules.

First, BTA belief is factive:

(1) $\mathbf{Bel}\, A \rightarrow A$

The reason is that, no matter what structural rules are assumed, provable consecutions $X \vdash A$ have the following property: If v is a Boolean valuation such that every formula B in X is true with respect to v, then A is true with respect to v as well. This may be easily demonstrated by induction on the complexity of proofs.

Second, BTA belief (in every Γ-state) is closed under Γ-consequence: If $A \vdash B$ is Γ-provable, then $\mathbf{Bel}\, A \rightarrow \mathbf{Bel}\, B$ is Γ-valid.[9] This is a straightforward consequence of the admissibility of (Cut).

Third, BTA belief is closed under "∧ elimination" and "∨ introduction":

(2) $\mathbf{Bel}(A \wedge B) \rightarrow (\mathbf{Bel}\, A \wedge \mathbf{Bel}\, B)$

(3) $(\mathbf{Bel}\, A \vee \mathbf{Bel}\, B) \rightarrow \mathbf{Bel}(A \vee B)$

This is a trivial consequence of the rules (∧E) and (∨I). Note, however, that BTA belief is not necessarily closed under "∧ introduction", since the converse of (2), *i.e.*

(4) $(\mathbf{Bel}\, A \wedge \mathbf{Bel}\, B) \rightarrow \mathbf{Bel}(A \wedge B)$

is not universally valid. For example, consider an epistemic state $s = (X, \Gamma, V)$, where Γ contains only (CI), $X = p\ ;\ q$ and $V(p) = V(q) = 1$. Obviously,

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[9] Hence, **Bel** may be read as *implicit* belief. Epistemic states may be extended by including a syntactic filter (such as an awareness set), but we shall not do so in the present paper.

Bel p and **Bel** q. However, there is no substructure of X that entails $p \wedge q$: the consecutions $p \vdash p \wedge q$, $q \vdash p \wedge q$ and $p \,;\, q \vdash p \wedge q$ are all clearly invalid.[10] The converse of (3) is not valid either. To see this, it is sufficient to consider a state s such that $X = p \vee q$ and Γ is empty. It is clear that neither $p \vee q \vdash p$, nor $p \vee q \vdash q$ are provable using the empty set of structural rules: there are Γ-countermodels to both consecutions.

Closure under modus ponens

(5) $\quad$ **Bel**$(A \rightarrow B) \rightarrow ($**Bel** $A \rightarrow$ **Bel** $B)$

is not universally valid either. Consider an epistemic state $s = (X, \Gamma, V)$, where Γ contains only (CI), $X = (p \rightarrow q) \,;\, (r \,;\, p)$ and $V(p) = V(q) = V(r) = 1$. Obviously, **Bel** $(p \rightarrow q)$ and **Bel** p. However, there is no substructure of X that entails q. We will provide a countermodel to $(p \rightarrow q) \,;\, (r \,;\, q) \vdash q$. (The reader may provide countermodels to consecutions with other substructures of X as an exercise.) Let $W = \{x, y, z\}$. As usual, let C and $\leq$ be identity on W. Assume that $Rxxy$ and $Ryyz$. Now let $E(x, r) = E(x, p) = 1$. Hence, $E(y, r \,;\, p) = 1$. Moreover, let $E(y, p) = 0$. Consequently, $E(y, p \rightarrow q) = 1$. Therefore, $E(z, (p \rightarrow q) \,;\, (r \,;\, p)) = 1$. But nothing prevents us from having $E(z, q) = 0$.

In general, consider the following epistemic closure schema:

(EC) If $A_1 \wedge \ldots \wedge A_n \rightarrow B$ is universally valid, then **Bel** $A_1 \wedge \ldots \wedge$ **Bel** $A_n \rightarrow$ is universally valid.

It is clear that only quite special cases of (EC) are true. For example, if $n = 1$ and $A_1 \vdash B$ is provable without any special structural rules. The latter condition is essential: many consecutions $A \vdash B$, where $A \rightarrow B$ is a propositional tautology, are *not* provable without recourse to specific structural rules. In fact, to achieve this was the point of introducing substructural logics.

Moreover, as the failure of closure under "$\wedge$ introduction" (4) demonstrates, (EC) does not hold if $n = 2$, even if $A_1 \wedge A_2 \vdash B$ *is* provable with-

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[10] A countermodel to the first consecution: Let W be a singleton consisting of x, let C and $\leq$ be identity on W and let $Rxxx$. Obviously, this is a (CI)-frame. Moreover, let $E(x, p) = 1$ and $E(x, q) = 0$. (There is a similar countermodel to the second consecution.) A countermodel to the third consecution: Let W be $\{x, y\}$, let C and $\leq$ be identity on W and let $Rxxy$. Moreover, let $E(x, p) = 1$, $E(x, q) = 1$ and $E(y, p) = 0$. Obviously, $E(y, p \,;\, q) = 1$, but $E(y, p \wedge q) = 0$.

out using any structural rules. In addition, it is plain that propositional tautologies are not universally BTA believed either: for example, $q \vee \neg q$ is a propositional tautology, but it is sufficient to consider a state s where $X = p$. It is plain that $p \vdash q \vee \neg q$ is not provable by using *every* Γ. To sum up, BTA belief does not suffer from many of the notorious omniscience properties.

5. Conclusion

We have outlined a simple formal model of belief that *i*) is based on true assumptions, *ii*) does not suffer from the usual omniscience properties. Moreover, the present framework is rather general: one may concentrate on various types of bodies of information (sets, multisets, lists etc.) and consequence relations (monotonic as well as nonmonotonic).

However, this paper is only an outline of a broader project. Many paths of future research are open. First, this paper does not discuss the problem of *axiomatisation* of the set of universally valid L_1-formulas. A related open problem is proving *correspondence results* for various epistemic formulas. Second, one may attempt to combine the substructural approach with the usual epistemic Kripke semantics and, in addition, to provide multi-agent versions. It is also possible to interpret several extended epistemic languages (possibly containing dynamic or group-epistemic operators) in these combined models. Finally, it is much desired to elaborate the present framework so that it could handle the familiar introspection properties (and formulas with iterated epistemic operators in general). However, these investigations are left for another occasion.

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