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Intuition and the End of All –Isms

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ABSTRACT: In my paper, some of the most influential *-isms* in the philosophy of mathematics are discussed with respect to their attitude to intuition. By the end of the all -isms, at first, their tendency to arrive eventually at just the opposite of their previously proclaimed principle is meant. The positive significance to the given tag line is connected with as a simple observation (due to both William James and Wittgenstein) that most of the -isms are justifiable if treated as practical attitudes rather than theoretical systems. Accordingly, intuition's role will be twofold: first, as a reference point with respect to which the given -isms were portrayed as turning into their very opposites; and, second, as the focal point to which all of them might be seen as contributing to intuition's pragmatic reading. Along these lines, the path of intuition might be transformed from an epistemological Calvary—or the path of despair, to use Hegel's words from the beginning of his *Phenomenology* in which one particular theory is replaced by another which is itself later replaced, etc.—into the path of progress in which some traditional dilemmas such as that between mathematical realism and nominalism are solved.

KEYWORDS: Brouwer – formalism – Frege – Hegel – intuition – intuitionism – logicism – phenomenology – pragmatism – philosophy of mathematics.

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1. Introduction

There is a famous remark by Albert Einstein (1998, 890) to the effect that any two -isms can be made the same if they are articulated properly. He made this remark in the context of some correspondence on scientific realism showing, furthermore, some tendencies towards the *pragmatic* standpoint in physics.

In my paper, I would like—as a kind of *dialectical* exercise—to adopt a similar attitude with respect to the main foundational streams in the philosophy of mathematics, including the doctrines of formalism, logicism, structuralism and intuitionism. The concept of *intuition*—to which all these doctrines refer, both in a positive and a negative way—will serve as the focal point upon which this exercise can be performed and eventually be brought to a pragmatic ending. The desirability of such an ending will be another point of my paper.

In general, I will not proceed by way of disambiguation but *phenomenologically*—in Hegel's sense of the word—which is to say, I will follow the given development in the philosophy of mathematics and let the given dialectic simply decide for us what an *intuition* might be.

2. Pure intuition

Let me start with some general remarks on *intuition*. It is a historical fact, which has been amply discussed in the literature, that the concept of intuition in the philosophy of mathematics as well as in philosophy proper has been used in *ambiguous* and often incompatible ways. Charles Parsons' (2009) book *Mathematical Thought and its Objects*, among others, might serve as a reference point proving that intuition has been treated as being of both of an empirical and an intellectual origin, receptive and spontaneous, subjective and objective, *de re* and *de dicto*, irrefutable and fallible, etc. In the end, the most stable property expected from intuition seems to be its *immediacy* going back to its origin in the verb "intueri", "to gaze at". Intuition and the knowledge based on it is thus typically contrasted with knowledge preceded by an inference as a kind of *mediation* and the prospective source of its instability. Let us take these general expectations—immediacy and reliability—as our starting point.

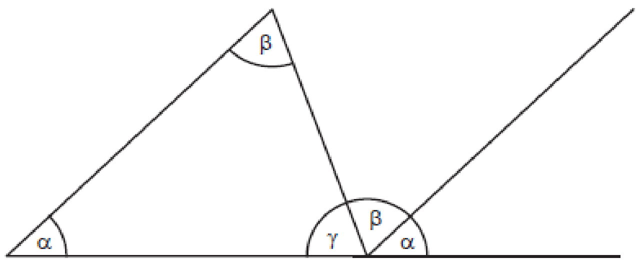
In mathematics, such a general attitude to intuition has an important precedent in antiquity where the demonstrative, *direct* methods of geometrically grounded mathematics had been contrasted with the *indirect*, dialectical methods of logic. As von Fritz (1971), Hintikka (1974) and others have argued, mathematics in antiquity was a science of *epagogic*—i.e. inductive—method as opposed to the *apagogic*—deductive—methods of the dialectic. Among the latter, the indirect proof was the most visible one used by both the rhetorician as well as the Eleatic sophist to justify some counterintuitive and unreliable claims such as that there is no motion, etc. As Grattan-Guinness (2000, 17) has noted, Kant in his controversial separation of mathematics from logic had only been following this old trail despite the existing tendencies to treat both, mathematics and logic, as sciences of the *analytic* method.

In the light of this, it is understandable why Kant's concept of intuition (*Anschauung*) is basically of *sensuous* origin. By intuition, basically, a representation of an object of our senses, is meant:

> In whatever way and through whatever means a cognition may relate to objects, that through which it relates immediately to them, and at which all thought as a means is directed as an end, is intuition. This, however, takes place only insofar as the object is given to us; but this in turn, is possible only of it affects the mind in a certain way. The capacity (receptivity) to acquire representations through the way in which we are affected by objects is called sensibility. [...] all thought, whether straightaway or through a detour, must ultimately be related to intuitions, thus, in our case, to sensibility, since there is no other way in which objects can be given to us. (Kant 1998, A19/B33)

As a result, in order to justify some sentence, no matter whether of empirical or mathematical origin—e.g., that the sum of the angles in a triangle equals two right angles—one has to intuit something, such as a particular triangle, through our *senses*. As Kant says referring to this very example (the only one he, in fact, gives besides the infamous 7 + 5 = 12), the mere concept or verbal definition of the given object is not enough. Contrary to the empirical context, though, in mathematics the given intuition has an *apodictic* power of the original epagogic method: The demonstration carried out for one particular geometrical figure (such as the given triangle, see picture) justifies the validity of the given claim for *all* the figures of the given form, and does so not only with some *probability* but with the utmost certainty.

In order to differentiate the empirical, probabilistic, from the mathematical, apodictic, induction Kant introduces the concept of so-called *pure intuition* in which the given demonstration is executed. By this very move, as might be expected, he did not resolve the unclear situation of intuition in mathematics. In fact, he made it worse to the extent



that, on the one hand, all the subsequent foundational doctrines, including formalism, structuralism, constructivism and intuitionism, but also logicism, conventionalism, axiomatism, etc., took his concept of intuition for granted, both as their basis and as their target, without, on the other hand, agreeing on at least some of its features as mentioned above. This leaves us, again, with the *immediacy* and *reliability* of pure intuition as something one can at least start with.

3. The logicist interlude

It was undoubtedly the phenomenon of *Non-Euclidian geometries* that made Kant's own example of mathematical justification (the sum of a triangle's angles) spurious: The given demonstration depends heavily on the validity of *Euclid's Parallel Postulate* and is thus *mediated* by it. Similar findings had gradually undermined the idea of spatial intuition's apodictic power and led, as their first fruits, to Poincaré's and Hilbert's conventionalism. Long before this, however, the indisputable success of Leibniz's and Newton's idea of calculus was confronted with its most blatant failures stemming from the uncontrolled employment of some "intuitive"—

both spatial and temporal—notions. This led to calculus' gradual reform which proclaimed all the intuitive references unreliable: That is why Lagrange limited himself to a purely formalistic justification of the calculus and why we shall not find any pictures in Cauchy's textbook.

In the next step of this reform, Bolzano and Frege take the overall counter-intuitionist attitude even further, claiming not only the general *unreliability* of intuition but its complete *uselessness*. The result is the doctrine of *logicism* according to which one can ground the whole arithmetic on logic and on logic only without any further reference to intuition. In the following, I take the case of logicism as the designated one not only because Frege phrased his anti-Kantian agenda in very Kantian terms—as the opposition between the intuition and the concept—but because, to some extent, by being the founder of formal logic in its second-order predicate form he provides an agenda for both the doctrine of formalism and of structuralism. In his *Foundations of Arithmetic* (Frege 1884, § 26), he even plays with the idea that intuitions and, in the end, even particular objects are purely subjective or not communicable and gives, as an example, the projective geometry where the intuition of point can be replaced by an intuition of line, etc., without changing the validity of the objectively valid laws such as that two points determine one line. But this looks rather like a slip of the pen according to Frege's own standards.

In his main counter-intuitionistic attack, Frege, as always careful in his foundational claims, does not phrase the uselessness of intuition as some obvious fact but phrases it explicitly as a promising *hypothesis* to be tested in his *Begriffsschrift* (Frege 1879, IV)—i.e. in the script wholly based on *concepts* thus avoiding hidden references to intuition. Because his foundational interests lie in arithmetic and in arithmetic only, he specifies his target explicitly as the Kantian *intuition of time* (Frege 1884, § 91). And he is quite explicit about what he means by that: Namely the dependency of proofs on the fact that numbers and arithmetical concepts have been introduced *recursively*, i.e. in a way in which the existence of objects and the validity of truths introduced "later", such as 5 or 7 + 5 = 12, depends on the objects and truths introduced "sooner".[2]

2 See Frege's critique of Grassmann in Frege (1884, § 6). For a detailed account of this point see my papers Kolman (2015), (2007). The rest of this section is significantly based on these papers.

Recursive definitions, quite common in arithmetic and, one would say, even intrinsic to it, are unacceptable for Frege for purely semantic reasons: The recursive definition of function *f* in which, first, the value for f(0) is set and, then, the value of f(x+1) is introduced by reference to the already set value of f(x), seems to talk about the object *f* sooner than it was definitely introduced. According to Dedekind and Frege, the only standard definition is the *explicit one* in which arithmetical concepts are not introduced *in steps*, as a sequence 1, 2, 3, 4, etc., but *at once* by means of a *single* formula Z(x). To achieve this, one must eliminate the "etc." clause from the recursive formations which Frege was able to do already in his *Begriffsschrift* by using the logic of the second order, particularly the definition of closure:

$$Z(x) := (\forall X)(X(1) \wedge (\forall x)(X(x) \rightarrow X(x+1)) \rightarrow X(x)).$$

In his own words, it was this success which convinced him of the viability of the logicist *hypothesis*.

As we now know, despite the original optimism, the logicist definitions have failed. The reason for this, though, is not the emergence of Russell's paradox, which, as the neo-logicists have shown, is eliminable anyway, but the very nature of *second-order logic*. If they are about to work properly as definitions of closure, the second-order replacements of recursive definitions cannot do without a supposition that there is an *infinite set* in the range of the second-order variable—the set must be *exactly* that of the natural numbers or, at least, of their structure, otherwise some unwanted objects might get into it. This is, in fact, what happens to the first-order Peano arithmetic within its so-called non-standard models.

Now, all the attempts to vindicate the existence of an infinite set by logistically acceptable means, such as Bolzano's (1851, § 13) and Dedekind's (1888, theorem 66) "pure" constructions, prove very graphically that there is no simpler way of introducing infinity than the original recursive, i.e. "intuitive" path, of which the sequence 1, 2, 3, 4, … is a model case. What's more, Frege's and Dedekind's attempts to eliminate the intuitive reasoning by focusing on axiomatic—i.e. formula-producing-systems—completely left aside the fact that the definition of a theorem or derivation proceeds by way of *recursion*. The attempt to eliminate "etc." and all the particular operations with artifacts such as pebbles or abacus from scientific arithmetic, leaving them to what Frege calls "Kleinkinderzahlen", is doomed to a vicious circle from the very beginning.

4. Three lessons

The lesson from the failure of logicism that is relevant here seems to be twofold:

(1) As for the first part, logicism concluded from intuition's unreliability or from the unreliability of the "gaze" that the alleged reliability of mathematics and knowledge in general must come from the other part of the Kantian distinction, namely from concepts of language. The original tension between clear-sighted intuition and blind symbols is thus turned *upside-down*. Blind symbols and concepts are now the only ones seeing, and intuition is now the blind one because it is unreliable. But this turn did not work. The alleged reliability of logic betrayed Frege at the very beginning, with principles he took not only for being true, but even for being true on analytic grounds.

(2) The second part starts with the supposition that one can do without intuition in the sense of leaving its manifestations to the pre-scientific or psychological level of pre-theoretical counting or drawing diagrams. In any mathematical science worthy of its name, one shall deal only with concepts. In the end, though, the concepts turned out to be not only unreliable but dependent in their goal on the recursive, i.e. intuitive definitions. Thus, ironically, all the remedies suggested by Frege or his followers, particularly type theory, consist in the employment of constructive principles, which is at blatant variance with the original anti-Kantian approach.

Now, based on this two-part lesson, it seems that one might feel compelled to adopt one of the following attitudes toward the logicist failure as far as intuition's reliability is concerned:

(1) First, there is the *attitude of Brouwer* according to which original intuition is reliable enough—one must only keep it sufficiently apart from the blind reasoning of logic.

(2) The second *attitude is that of the early Hilbert* who accused Frege and his logic in general of not being formal or conceptual enough in that they still give, even in their anti-intuitionistic attitude, too much room to intuition by simply presupposing that there is something beyond their formulas which these try to express.

Interestingly, in their foundational endeavors neither Hilbert nor Brouwer fare much better than Frege, ending up with the very opposite of what they promised to achieve.

Brouwer's appeal to a more intuitive mathematics that does not depend on the linguistic schemata but, instead, is anchored in the constructive decision of the *creating subject*, led famously to theorems which almost nobody—including Brouwer's followers—took to be intuitive or even true. From the other side, the variety and artfulness of ways in which Brouwer tried to refute the classical theorems, including such absolute "certainties" as the *principle of the excluded middle*, not only gave rise to the one and only split in modern mathematics but inspired Wittgenstein—after attending one of Brouwer's Vienna Lectures—to enter the second period of his thinking, one characterized by a belief in the plurality of language games as opposed to the intuitively given discourse.

As for Hilbert, his first version of formalism and his attitude to intuition started with the explicit idea that mathematics' certainty consists in concrete but blind symbols and their finite organizations in formulas and formal derivations. The question whether these symbols refer to something—e.g., to infinite entities as Cantor suggested—was bracketed not for being unjustified or unscientific, but rather for being *irrelevant* as far as the issue of foundations is concerned. Gradually, this cautious approach had become an intrinsic one adopting noticeable *transcendental* features: Since the roots of any knowledge are to be identified with a finite (or finitely describable) system of rules and axioms, and finite deductions from them, the certainty of them is also the certainty of given intuition, which, similar to the pure intuition of Kant, is thus not purely empirical but has apodictic features. This is the so-called "finite Einstellung".[3]

In the light of this, one can say that Hilbert and Brouwer represent, in the philosophy of mathematics, certain kinds of *antithetical* positions

3 I elaborated on this point in my paper Kolman (2009).

reminding one of the early chapters of Hegel's *Phenomenology of Spirit*. Analogously to its starting position of *immediately* given and certain knowledge, which—compared to its own standards—turns out to be the most general and mediated, Brouwer starts with the self-certainty of the given intuition only to end up with the most uncertain and counter-intuitive results. In the next stage, one decides with Hilbert to eliminate the reference to intuition and its object in favor of a meaningless language so as to be forced to acknowledge a new kind of intuition dealing with linguistic artefacts. These examples, of course, make sense only as a part of a bigger story that I have tried to develop elsewhere.[4]

Its lesson, obviously, is not historical but rather *dialectical*. Namely, that there is another, third, secret part of the lesson to be taken from the original logicist failure. And this consists in the conclusion that intuition—even in its pure form—does not have to be *immediate* and *reliable* in the absolute sense of the word. In the same sense in which I do not doubt that this is my hand (to quote G. E. Moore while raising my hand), I will not doubt that the sum of all angles in a triangle equals two right angles. This is not to say that, e.g., by empirical measuring, different results cannot come about, but that they are not typically treated as *counter-instances* to the given claims but as failures to be ignored. And this is what we mean by the given sentences to be *a priori*: That we treat them as irrefutable by standard singular experience because this standard experience—or let us say, with Wittgenstein, the whole stage on which it is played—is defined by their *stability*. But this stability is only a relative one and might be shaken by some drastic change in the situation, e.g., if some secret surgery were performed on me or in the need to measure cosmic distances and times.

5. Pragmatic turn

The *relative* concept of a priori, at which we have just arrived, has been commonly and prominently advocated by, e.g., C. I. Lewis and Ludwig Wittgenstein. What is not so common is the corresponding readjustment of *intuition* to these philosophical needs, the concept alone being abandoned

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4 See my book Kolman (2016a).

rather than justified by modern philosophy, as, for that matter, the development of continental phenomenology and its transition from Husserl to Heidegger testifies. To make the philosophy of mathematics up-to-date, though, one does not have to leave intuition aside as something contradictory and obsolete. What one needs to do is to learn from the development of failed attempts at making the founding principle of mathematics explicit. This is the moment where the phenomenological method of Hegel enters the stage.

As for arithmetic, what has been seen is a repeated pattern of the rebirth of the *constructive* even in the most abstract disciplines of mathematics such as set theory and logic. In light of this, even the "revolution" of Brouwer does not seem to be such a radical break with the whole development but instead represents an explicit acknowledgment of their tacit preconditions to which the systematic use of constructive principles such as *transfinite induction* or situation-dependent formations such as *diagonalization* belong. This does not need to be read as a defense of *constructivism* but simply as an alert that many of the theoretical questions have a *practical* dimension which cannot be eliminated from the foundational debates. As a result, we should enrich the concept of intuition by this practical aspect. Such an adjustment is, in fact, in accord with Kant's original conception of pure intuition which is always explicitly connected to *constructions*—i.e. to doing something—in space and time.

In the realm of geometry, e.g., by claiming that two different lines orthogonal to the third line cannot intersect in any *possible* prolongation, one can mean neither an empirical nor a purely theoretical possibility but a practical and normative one of prolongations that are "good enough" or "acceptable." The mathematicians' talk about the intersection in infinity is thus only a theoretical abbreviation for this practical certainty which, in the context of cosmic distances, loses its original sense. So the discovery of non-Euclidian geometries and their successful applications in physics does not count as an absolute refutation of Euclidian geometry but only as a kind of *proto-theoretical* impulse to revise it with respect to the given context.

Drawing on Lorenzen's work,[5] Stekeler (2008) in *Formen der Anschauung* elaborates on this basic approach to Euclidian geometry starting with the postulates from which the quality of *rectangular solids* (or *blocks*) and

[5] Particularly Lorenzen (1984).

wedges should be measured.[6] A block is defined as a solid fulfilling the following principles:

(1) It has 6 surfaces, 12 edges and 8 corners,

(2) the surface of a block fits on the surface of its copy and on the surface opposite to it thus forming a new block of a bigger size,

(3) through the given inner point a surface of the given block can be uniquely cut up into 4 smaller blocks,

(4) two not necessarily congruent blocks can be brought into (partial) overlap in two arbitrarily marked places on their surfaces,

(5) two blocks lying on the surface of the third block are overlapping already in the case that they touch in two places of their opposite edges,

(6) through the diagonally opposite edges of a given block there is only one diagonal plane cut that divides the block into two rectangular wedges that are copies of each other,

(7) by removing or adding the congruent bodies to a congruent body at the same place the congruent bodies are obtained,

(8) for every two edges of two blocks there is a natural number *n* such that the edge of the one block after *n* applications exceeds the edge of the other one.

These postulates are obviously neither axioms in the traditional sense of self-evidently true sentences nor in Hilbert's modern sense of implicit definitions. They are *material norms* defining the given concept by recourse to the pregiven practice of forming the solids and assessing the quality of their form to the extent that it is the very possibility of this practice that guarantees that these postulates are (in)dependent and consistent. By their *completeness*, Stekeler means that they are sufficient to found classical Euclidian geometry in an inferentially-holistic way, forming what is known as its standard model. The basic geometric concepts such as *flat surface*, *straight line* or *orthogonality*—or theorems about them—are taken

6 See also my review Kolman (2011).

to be the simple (material) consequences of the postulates: plane is the surface that fits on a block, straight is the line fitting on the edge of a block and orthogonal is the angle formed by two intersecting edges of a block. The parallel postulate which is not (formally) deducible from the rest of Euclid's or Hilbert's axioms is a material consequence of the postulates (6) and (8).

Similarly, in *arithmetic* one can understand *Peano axiom*s as material norms expressing the truth about working within the underlying calculi and not as some a priori given truth about some independently given objects. I will come to this in the next, final section. What matters now is that, along these lines, the Kantian concept of pure intuition can be reconstructed in a way which does not have to follow all the details of the Kantian corpus and yet will still remain true to the original idea of his philosophy. This amounts, in the end, to the general insight that the differences one makes do not exist here simply *in itself*, but always *for us* as cognitive subjects. This reading includes also the later rectification of Hegel, by which the a priori structures of reason cannot be interpreted as belonging to the *privacy* of a subject's mind—as some of Kant's followers presumed—but in the joint practice of our orientation in space and time. The general message is simple: One cannot ground any knowledge by merely *looking at* things. This is not only because every act of looking is theoretically charged, but because it is substantially clothed in social agency. The following specification of intuition is given by Stekeler (Unpublished):

> *Anschauung* stands for any possibly *conceptually articulated reference to some object or event in real perception*—such that the same object can or could be perceived by others as well. *Pure intuition* is a label for the mere form of such an objective reference to objects of perception—including the corresponding spatial and temporal transformations of perspectives if there are different observers at different places or if we refer to the same object or event from different times.

Such a practically and socially articulated intuition cannot be infallible simply because I, as the cognitive subject, can never be the absolute guarantee of the corresponding truth. But this feature, as Wittgenstein (and Hegel) have taught us, *makes such an intuition as a prospective basis of knowledge something that is quite impossible in its immediate and infallible form.*

6. Gödel theorems

As in philosophy proper, in the philosophy of mathematics this lesson has been learned the hard way to the extent that there is a kind of official narrative in which Hilbert is the last hero of the old times and Gödel theorems are the living memorial of the last hero's fall. But the situation is already, in fact, much more simply captured by Bernays' laconic remark from his commentary to Hilbert's (1935, 210) collective works:

> it has turned out that in the realm of meta-mathematical reasoning the possibility of a mistake is particularly great.

And this is simply because, e.g., the claims about derivability (and non-derivability) of some figure—despite their being about specific symbols—obviously exceed the "here and now" of the given intuition and point to something which is mediated by its very form. One might call this form the "pure intuition".

Gödel theorems are, in this very sense, not the end of Hilbert's *finitist* approach but rather its correction underlining the mediated and practical nature of our experience. Following Hegel, one can call this feature "infinite" not in the sense that it leads us beyond our "finite" experience—which is the sign of Hegel's famous concept of "bad" infinity—but that it leads beyond its too narrow delimitation: Gödel's unprovable yet true sentence still has to be proven to be true but not in the overly narrow context of Hilbert's methods.[7] What I am aiming at is that Hegel's distinction between bad and true infinity might be fruitfully applied to the concept of intuition and its development with respect to the phenomenon of Gödel theorems.

As his unpublished papers show, Gödel (1995, 310) himself oscillated between the following readings of his results:

(1) "there exist *absolutely* unsolvable diophantine problems [...], where the epithet 'absolutely' means that they would be undecidable, not just within some particular axiomatic system, but by *any* mathematical proof the human mind can conceive" (the diophantine problems

[7] I discuss the mathematical and logical relevance of Hegel's concept of "bad infinity" in my paper Kolman (2016b).

are of the so-called Goldbach type, i.e., of the form $(\forall x)A(x)$, where $A(x)$ is a decidable property of numbers),

(2) "mathematics is incompletable in this sense, that its evident axioms can never be comprised in a finite rule, that is to say, the human mind (even within the realm of pure mathematics) infinitely surpasses the powers of any finite machine."

Accordingly, Gödel might be seen as adjusting Hilbert's narrow, finite concept of intuition by two alternatives, the bad and the true one. The first one, the most popular among the working mathematicians and general public, amounts to claiming that Hilbert's methods do not exhaust "all our mathematical intuitions" we have about the subject. This is, of course, the radical inversion of the original concept of "Anschauung", transforming it from a matter of direct insight based on the sensuously given (e.g., the signs of language) into some mysterious voice from the grave lying beyond our senses (or our language). Such an intuition, of course, tends to be fallible almost by definition.

The other concept of intuition which is in accord with our previous practical delimitations does not lead us *beyond* our language (or the sensuous data in general) but merely beyond its too narrow understanding of a mere artifact. Besides the visual—intuitive—form of the spoken or written signs there is something which gives them life; namely, their use within the whole of human practice that they belong to. In this reading, Gödel theorems might be looked at as elaborating on this distinction between the sign and its use, or between the intuition in the narrower (merely sensuous) and broader (practical) sense.

Following the line of thought indicated in the geometrical case, with axioms interpreted as material norms embedded into the practice of forming solids, I suggest doing the same in the case of arithmetic with axioms interpreted as norms embedded into the practice of calculating and measuring. The continuity between Hilbert's and Gödel's approach to intuition will be secured by replacing the standard difference between the axiomatic theory and its model—which simply copies the ontological difference between the sign and the external object this sign refers to—by two kinds of axiomatic systems and the corresponding concepts of consequence: strongly effective or *full-formal* ($\vdash$) and the more liberal or *semi-formal*

($\models$). Going back to Schütte (1960), both these differences were developed by Lorenzen (1962) in his *Metamathematik* and might be specified as follows:[8]

Full-formal arithmetic, like the arithmetic of Peano, is arithmetic *in the narrower sense* and deals with schematically or mechanically given and controllable axioms and rules. Semi-formal arithmetic or *the* arithmetic *proper* employs—in accord with the infinite nature of the number series 1, 2, 3, …—rules with infinitely many premises, particularly the (ω)-rule A(1), A(2), A(3), etc. ⇒ (∀x)A(x), which is nothing else than the instance of the so-called *semantic definition of truth*. Hence, the significance of semi-formalism is to make us think of semantic definitions as special (more generously conceived) systems of rules (proof systems) which—starting with some elementary sentences—evaluate the complex ones by *exactly one* of *two* truth values.

It is a known fact that the intuitionists and some constructivists (including Lorenzen, but not, e.g., Weyl) question the completeness of this evaluation, arguing that the existence of concrete strategies for proving or refuting every A(N) doesn't entail the existence of a general strategy for A(x). Consequently, a decision must be made whether the infinite vehicles of truth as (ω) should be referred to as rules (1) only in the case when we positively know that all their premises are true, i.e. when we have at our disposal some general strategy for proving all of them at once or, (2) more liberally, if we know somehow that all their premises are positively true or false. The general distinction between the constructive and classical methods in arithmetic is based on this. Now, if one leaves, like, e.g., Lorenzen and Bishop, the concept of effective procedure or proof to a large extent open and does not tie it, like, e.g., Goodstein and Markov, to the concept of the Turing machine, there is still room for an effective, yet liberal enough *semantics* (semi-formal system) and a strongly effective syntax or *axiomatics* (full-formal system). Hence, the constructivist reading does not necessarily wipe out the differences between the proof and truth, as, e.g., Brouwer's mentalism or Wittgenstein's verificationism seem to. And this, in the following way, is where the true concept of intuition comes from:

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8 The argument given here, and the rest of this section, is based on my paper Kolman (2009).

Gödel theorem affects *only* the full-formal systems, because their schematic nature makes it possible to devise a general meta-strategy for constructing true arithmetical sentences not provable in them. The unprovable sentence of Gödel is of the form $(\forall x)A(x)$, where $A(x)$ is a decidable property of numbers. Now, Gödel's argument shows that this decision is done already by Peano axioms in the sense that all the instances $A(N)$ are deducible and, hence, set as true. So, with Gödel's proof we have a general strategy for proving all the premises $A(N)$ at once, which makes the critical unprovable sentence $(\forall x)A(x)$ constructively true, i.e. provable by means of the (ω)-rule interpreted *constructively*. As a result, there is an intuitive way that transcends the methods of Hilbert's "finite attitude" and that allows us to see why Gödel's theorems did not destroy but instead *refined* Hilbert's finitist—and intuitive—approach in the suggested semi-formal way.

7. Conclusion

In my paper, some of the most influential -isms in the philosophy of mathematics have been first discussed with respect to their attitude to intuition. By the end of the all -isms, their tendency to arrive eventually at just the opposite of their previously proclaimed principle might be meant. But there is a deeper significance to this tag line connected with the suggested pragmatic closure of the paper: This was not meant as a replacement of the given -ism by another one (such as constructivism or pragmatism), but as a simple observation (due to both William James and Wittgenstein) that most of the -isms are justifiable if treated as practical attitudes rather than theoretical systems. Accordingly, intuition's role was twofold: first, as a reference point with respect to which the given -isms were portrayed as turning into their very opposites; and, second, as the focal point to which all of them might be seen as contributing to intuition's pragmatic reading.

I tried to sketch how, along these lines, the path of intuition might be transformed from an epistemological Calvary—or the path of despair, to use Hegel's words from the beginning of his Phenomenology in which one particular theory is replaced by another which is itself later replaced, etc.—into the path of progress in which some traditional dilemmas such as that between mathematical realism and nominalism are solved. This is in accord with Hegel's own intentions and his general idea to look at the desperate—

or negative—nature of knowledge in a cautiously positive way: “this path is the conscious insight into the untruth of knowing as it appears, a knowing for which that which is the most real is rather in truth only the unrealized concept” (Hegel 2018, 52). On such a path, though, there are no signposts or a particular -ism to be mechanically followed and, accordingly, new problems and dilemmas are arising simply because, by “practical”, a lot of things can be meant. This has been shown, e.g., by the case of the word “effective” or “effectively calculable” in the context of theorems such as the Church-Turing thesis and the subsequent development of constructive mathematics.

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