Organon F 21 (4) 2014: 492-515

Quantificational Accounts of Logical Consequence III. The Model-Theoretic Account: Quantificational Approach Triumphant?[1]

LADISLAV KOREŇ

Department of Philosophy and Social Sciences. Faculty of Arts. University of Hradec Králové
Náměstí Svobody 331. 500 03 Hradec Králové. Czech Republic
ladislav.koren@uhk.cz

RECEIVED: 06-09-2013 ▪ ACCEPTED: 21-08-2014

ABSTRACT: This concluding study devoted to quantificational accounts of consequence and related logical properties deals with the model-theoretic account (MTA). In response to objections questioning its intuitive adequacy, it is argued that MTA does not aim to analyse "the" alleged intuitive notion of consequence, but aims to formally reconstruct one specific semantic account, according to which valid arguments preserve truth in virtue of their logico-semantic structure and irrespectively of particular semantic values of the non-logical vocabulary. So conceived, MTA is arguably superior to any other quantificational account, being based on a principled account of the semantic structure and the specific contribution of logical elements to it.

KEYWORDS: Logical consequence – models – quantificational accounts – validity.

1. Introduction

In my penultimate study (see Koreň 2014) on quantificational accounts of consequence, I discussed modern substitutional (Russell 1918/1919, 1919, Carnap 1937, Quine 1986) and interpretational (Tarski 1936) expla-

[1] My work on this study was supported by the IP project *Otázka relativismu ve filosofii a společenských vědách* at the Philosophical Faculty of the University of Hradec Králové.

nations of valid arguments as those that preserve truth under all admissible variations with respect to their non-logical vocabulary. I highlighted two difficulties for such explanations: they might overgenerate due to limited expressive means or due to assuming a fixed domain of quantification. Tarski-style interpretational accounts avoid the first difficulty but the second remains pertinent. Quine's update on the substitutional account hopes to avoid both problems, but it is restricted to first-order languages rich enough to embed elementary number theory.[2] I suggested two reasonable desiderata that might be imposed on appealing quantificational accounts: (1) logical properties/relations of sentences should persist under subtractions and expansions of the non-logical vocabulary; and (2) they should persist no matter what sequence of possible semantic values of appropriate types we assign to their non-logical elements, whatever possible domain of application those values may come from. I concluded by saying that the model-theoretic account (henceforth MTA) of logical consequence as *truth-preservation across all admissible set-theoretic interpretations* suggests itself as an advance in this respect, as it appears to be taylor-made to meet the two desiderata. Many logicians would agree that MTA is the most promising semantic approach to consequence currently on the market, not least because it provides rigorous explications of logical notions – relative to a principled account of the semantic behavior of certain traditionally distinguished logical operators – that makes room for mathematically tractable metatheoretical comparisons between the semantic and the deductive side of logic. That said, MTA has been subjected to vigorous criticism questioning its adequacy as an account of consequence. In particular, it has been claimed that it blatantly fails as an account of consequence, because it inevitably misses certain essential modal-epistemic characteristics of it (cf. Etchemendy 1990). In this concluding part of my explorations into the quantificational tradition, I discuss how MTA fares *vis-à-vis* the main philosophical objections in this direction, suggesting considerations that conspire together to provide a partial vindication of MTA.

[2] And, as I pointed out in Koreň (2014, 322), identity is not regarded as a logical primitive but as a defined predicate expressing indistinguishability with respect to all (*n*-adic) predicates of the object-language.

2. MTA

As is established practice, we will introduce the essentials of MTA by focusing on a first-order language L (with identity), containing denumerably many individual constants and variables, *n*-adic predicates and *n*-adic functors. With the syntax defined in the usual way via recursive definitions of the sets of *L-terms*, *L-formulas* and *L-sentences*, the idea of interpretation of L in L-structure $\mathfrak{R}$ is implemented by taking a non-empty set d as the domain of $\mathfrak{R}$, and assigning extensions of appropriate types (defined over d) to non-logical symbols, according to their categories: (a) an element $c^{\mathfrak{R}} \in d$ to each individual constant c; (b) an *n*-adic relation $P^{\mathfrak{R}} \subseteq d^n$ to each *n*-adic predicate P (for every $n \geq 1$); (c) an *n*-adic operation $f^{\mathfrak{R}}: d^n \rightarrow d$, to each *n*-adic function symbol f (for every $n \geq 1$). This amounts to a set-theoretic interpretation of L in $\mathfrak{R}$, usually represented as the ordered pair $\langle d, I \rangle$, with I being an interpretation-function accomplishing the job of (a),…,(c) above. On this basis, satisfaction of a formula of L in $\langle d, I \rangle$ by a variable-assignment (function assigning d-elements to individual variables) is defined by recursion on the logical complexity of the formula. As a limiting-case, then, truth of a sentence *A* (formula with no occurrences of variables free) in $\langle d, I \rangle$ is defined as its satisfaction by all variable-assignments of d-elements:[3]

$\mathfrak{R} \models A$ iff $\mathfrak{R}, s \models A$, for every variable-assignment s of d-elements,

where "$\mathfrak{R}, s \models A$" says that *A*, as interpreted in $\mathfrak{R}$, is satisfied by a variable-assignment s of d-elements. If we read "$\mathfrak{R} \models A$" as saying that $\mathfrak{R}$ is a *model* of *A*, we can finally define crucial model-theoretic notions as follows:

$\mathfrak{R}$ is a model of a set Γ of L-sentences iff $\mathfrak{R}$ is a model of each sentence in Γ.
A is valid iff every L-structure $\mathfrak{R}$ is a model of *A*.
A is a consequence of Γ iff every L-structure $\mathfrak{R}$ that is model of Γ is also a model of *A*.

The idea here is that whereas the semantics of L's logical constants is fixed (the same irrespectively of what admissible structure interprets L) via

[3] I pass over tedious details, the requisite machinery being contained in any standard mathematical logic book such as Hodges (1997) or Enderton (2001).

recursive clauses of the general satisfaction definition, non-logical symbols of L are allowed to pick out different extensions in L-structures. Importantly, however, possible interpretations of non-logical constants are in no way arbitrary but have to harmonize in principled ways with the semantics of logical operators. Assignments of semantic values to non-logical constants are to be such that, in cooperation with the fixed meanings of the logical operators, they suffice to determine the truth-values of L-sentences. The rationale for this requirement should be clear: admissible interpretations of non-logical vocabulary encapsulate as much and only as much information as is required to fix truth-values of L-sentences in accordance with the semantics of their logical operators. For instance, the truth-value of the universal formula $\forall x(Fx \rightarrow Gx)$ is going to depend on what domain d the universal quantifier ranges over, the quantifier being sensitive only to extensions of predicates "F" and "G" in that domain. In general, the Fregean idea of the semantic value of a non-logical constant *C* is precisely the idea of a truth-relevant feature of *C*, to which solely the logical operators are sensitive (which operate on the non-logical expressions of *C*'s type). An interpretation of *L* is then a systematic assignment of such truth-relevant features to its non-logical terms.[4]

Equipped with such semantic explanations and definitions based on them, MTA solves the problem of persistence-violation in Tarski's style – employing the method of satisfaction of formulas by variable-assignments of d-elements. Its comparative advantage *vis-à-vis* substitutional accounts is that it does not stand and fall with the expressive capacity of a language under consideration. And, by allowing domains to vary across set-theoretic interpretations, it avoids the problem of overgeneration that confronted interpretational accounts in the style of Tarski (1936). Consider a cardinality-sentence such as $\exists x \exists y \neg(x = y)$, which the interpretational account declares as logically true if it is true (if there are two or more objects), and as logically false if it is false (there being nothing to reinterpret in it, both quantifiers and "=" being regarded as fixed logical terms). However, it is not logically true (false) according to MTA, because it turns out false when interpreted in set-theoretic structures whose domains contain one object

4 A comparative advantage of verifying interpretations *vis-à-vis* verifying instances is that while there is an interpretation for every verifying instance of a formula, structures are more manageable, encapsulating truth-relevant features that may be shared by several instances.

(and true if interpreted in bigger domains). The same holds, *mutatis mutandis*, for other cardinality sentences of this type (for $n>1$), all of which have models as well as counter-models in the vast realm of set-theoretic structures. Consequently, also formalized versions of inferences of the following type (or zero-premise inferences with such cardinality sentences as conclusions)[5]

$$\frac{\text{There are at least } n \text{ objects}}{\text{There are at least } n\text{+1 objects}}$$

are not model-theoretically valid, since there is always a set-theoretic counter-model whose domain contains no more than *n* objects. Analogous considerations pertain to formalized versions of cardinality sentences of other types (e.g. those stating an upper bound on the size of the universe) or zero-premise arguments whose conclusions are such sentences.[6]

3. But is MTA really satisfactory?

Is the problem of persistence under all possible contractions and expansions of the quantifier-domain thereby solved? It may seem so, because logical properties now persist not just under all possible contractions and expansions of the non-logical vocabulary, but under all possible contractions and expansions of the domain of quantification, that is, no matter what domain of set-like size the individual variables range over. That said, the critic of MTA may retort that this glorious victory is pyrrhic, as it does not come for free. To use Quine's words, with model theory we are far away from "the modest bit of set theory" (viz. finite sets),[7] being committed to

---

5 Of the type: $\exists x_1 \ldots \exists x_n[\neg(x_1 = x_2) \wedge \ldots \wedge \neg(x_1 = x_n) \wedge \neg(x_2 = x_3) \wedge \ldots \wedge \neg(x_2 = x_n) \wedge \ldots \wedge \neg(x_{n-1} = x_n)]$, therefore $\exists x_1 \ldots \exists x_{n+1}[\neg(x_1 = x_2) \wedge \ldots \wedge \neg(x_1 = x_{n+1}) \wedge \neg(x_2 = x_3) \wedge \ldots \wedge \neg(x_2 = x_{n+1}) \wedge \ldots \wedge \neg(x_n = x_{n+1})]$.

6 Except for famous (or infamous) sentences such as $\exists x(x = x)$, whose natural reading is that *there is at least one object*, and which is satisfied in all first-order structures (as all have non-empty domains). Other problematic sentences are of the type $\exists x(x = a)$, $\exists x(Fx \vee \neg Fx)$, $\exists x(Fx \rightarrow Fx)$.

7 The modest bit of set theory that is needed in Quine's view to provide a substitutional account of logical properties for first-order languages rich enough to embed elementary number theory, which is provably equivalent to the set-theoretic account in the

"the universe of sets of a specifiable and unspecifiable size" (Quine 1986, 55). Now, this warning may or may not appeal to us, depending on whether we want to extend logic beyond first-order systems and whether we take the amount of "higher" set theory needed to build models for such languages to be sufficiently clear. But even if we set aside Quine's scruples, it might be argued that the trouble with the model-theoretic account is deeper. For isn't the problem of intrusion of substantive assumptions into logic still with us, this time reappearing on the level of meta-theory with its set-theoretical assumptions? The objection may be pressed as a version of Wittgenstein's challenge that we saw at work when we discussed Russell's substitutional account (cf. Koreň 2014): if logical relations and properties are to be construed as formal and topic-neutral, it would seem that they should not be contingent on substantial truths of whatever sort. However, MTA appears to make them contingent on substantive matters, this time in the form of specific background set-theoretic assumptions; so logical properties and relations are not distinguished from substantive (indeed, topic-specific) generalizations in terms of sets.

In this spirit, Etchemendy (1990) argued that the model-theoretic account of logical properties is guilty of a misguided reduction, precisely because possession of a logical property such as logical truth (or logical validity, in case of arguments) is equated to the truth of a certain set-theoretic generalization. To see what is at stake, consider a first-order sentence

$$S[a_1,..,a_1,\ A_1,..,A_n],$$

where $\{a_1,..,a_1\}$ is the set (possibly empty) of all its individual terms, and $\{A_1,..,A_n\}$ is the set of all its unary predicates (possibly empty). We shall assume that S does not contain any other non-logical constant. As Etchemendy notes, the model-theoretic account declares S as logically true just in case the following set-theoretic sentence holds (for $1 \leq i \leq n$):

$$(\forall d)\ (\forall x_i \in d)\ (\forall X_i \subseteq d)\ (x_1,..,x_1,\ X_1,..,X_n),$$

in which the non-logical constants are uniformly replaced by variables of fitting logical types (a_i/x_i ; A_i/X_i), and the result is universally closed with

model-theoretic style. Cf. Quine (1986, 56). See also my discussion of Quine's account in Koreň (2014, 319-323).

respect to all variables.[8] This set-theoretic sentence states that the matrix $(x_1,..,x_1, X_1,..,X_n)$ is satisfied with respect to all individual domains, whenever we assign elements of such domains to individual variables and subsets thereof to predicate variables. In effect, MTA tells us that:

> S remains true no matter what domain d we take S to talk about, no matter what individuals $x_1,..,x_1$ (from d) the terms $a_1,...,a_1$ respectively pick out and no matter what subsets $X_1,..,X_n$ (of d) the predicates $A_1,..,A_n$ respectively pick out.

This consideration applies, *mutatis mutandis*, also to the model-theoretic account of logical consequence as truth-preservation across all set-theoretic structures. Fully spelled out, what MTA states is that an L-sentence *A* logically follows from a set Γ of L-sentences just in case a certain set-theoretic sentence holds, namely one stating that every model of Γ is also a model of *A*.

At first blush, MTA is an obvious advance over interpretational accounts in the style of Tarski (1936), helping us to address the overgeneration problem due to the fixed domain. But Etchemendy thinks that this reductionist maneuver is still misguided as an analysis of logical properties, because it makes the logical status of a sentence (or argument) dependent on whether a substantive set-theoretic generalization holds, hence on extra-logical matters such as: whether there are sets, or how big (see Etchemendy 1990). For instance, if there were only finite sets, some substantive (e.g. cardinality) sentences should be declared logically true according to MTA, since there would not be enough (large) structures to show that they could fail to hold. Admittedly, the model-theorist might invoke the background set-theoretic axiom of infinity to ensure that there will be no shortage of sufficiently large sets (hence structures) to frame countermodels to such sentences. But the question then is why this manoeuvre is not on a par with the logicist postulation of the axiom of infinity that, many would agree, is suspect from the point of view of pure logic. Intuitively, such sentences are logically contingent and should remain so no matter whether sets are actually finite or infinite. Indeed, it would seem that a staunch finitist could consistently claim both that there are (or *can be*) only finite sets and that cardinality sentences holding in all finite mod-

---

[8] The variables are to be chosen so as to avoid potential clash with bound variables already present in S.

els are still not logically true (see Etchemendy 1990, 195). If this is intelligible, model-theoretic validity just cannot capture the essence of our intuitive notion of consequence. Or so Etchemendy argues, concluding that MTA fails to provide adequate conceptual analyses of logical properties.

Etchemendy pushes this line of objection yet further, maintaining that although the standard MTA does not overgenerate in the first-order case (i.e. it does not declare as logically true/valid sentences/arguments that are not intuitively logically true/valid; see Etchemendy 1990, 154), this is no more than a happy coincidence owing to expressive idiosyncrasies of first-order languages (that the account does not overgenerate he takes to be shown by a version of Kreiselian squeezing argument to be reviewed shortly).[9] In the higher-order case we do not have any guarantee that the model-theoretic account does not overgenerate, declaring as logically true (valid) sentences (arguments) that are intuitively not logically true (valid). In fact, Etchemendy argues that it overgenerates, since there is a second-order formalizable sentence CH, expressed purely in variables and logical symbols, that is true in all standard (full) structures[10] just in case the Continuum Hypothesis holds, and a sentence non-CH of a similar character that is true in all such structures just in case the hypothesis fails to hold.[11] So, depending on whether the hypothesis holds or not, either CH or non-CH is declared as logically true by the model-theoretic account for second-order logic. But isn't that weird, given that the hypothesis seems to express a rather substantive mathematical fact, if true? Once again, logical properties

---

9 As Blanchette (2000; 2001) points out, even this claim needs qualification, as the set of first-order validities (which, by the completeness theorem, is included in the set of purely logically provable first-order sentences) includes also *prima facie* logically contingent sentences of the type $\exists x(x = x)$, $\exists x(x = a)$, $\exists x(Fx \lor \neg Fx)$, $\exists x(Fx \rightarrow Fx)$.

10 The standard semantics of second-order logic works with full models interpreting second-order predicate variables as ranging over *all* sets of *n*-tuples of the domain over which first-order variables range. This does not hold for Henkin's non-standard semantics that does not force second-order variables to range over all such subsets.

11 The Continuum Hypothesis says that there is no set such that its cardinality is greater than the cardinality of the set of natural numbers but strictly smaller than the cardinality of the set of real numbers. If we let „$x > N$" and „$R \leq x$" to abbreviate second-order definable properties of *being of greater cardinality than the set of natural numbers* and *being of no smaller cardinality than the set of real numbers* respectively, CH can be captured thus: $\forall X(X > N \rightarrow R \leq X)$. Cf. Blanchette (2001, 128).

explicated *à la* MTA seem to be contaminated by apparently extra-logical matters (see Etchemendy 2008, 176-177).[12]

One's response to this kind of objections is going to depend on one's view of the nature of logic and its relation to mathematics. Several stories may be told here, but I favour the one in which mathematics (its set-theoretic branch) is utilized by logicians as a powerful modelling device, where modelling involves vital aspects of representation, idealization and abstraction.[13] Specifically, mathematical tools help us reconstruct correct reasoning in various domains, disregarding irrelevant features, while retaining and possibly sharpening features deemed central – and all this for theoretical and practical purposes at hand (cf. Priest 1999). This approach to logic seeks not only a proper balance – reflective equilibrium – between general theoretical principles, informal desiderata (e.g. a prioricity, necessity, formality) and intuitive judgements concerning validity, but the whole enterprise is open to revision and subject to the criteria of simplicity or economy, just as any other theory using mathematical models of real-world phenomena.[14]

Viewed from this perspective, as Shapiro (2005) points out, it makes little sense to say that MTA aims to "conceptually analyze" intuitive notions of logical truth or consequence – still less "the" intuitive notions of logical truth or consequence – when defining their idealized formal-mathematical counterparts (see also Priest 1995). Etchemendy's objection that MTA fails as a conceptual analysis is thus off the mark, as MTA does not aim to provide conceptual analyses (in whatever plausible sense this may have). Still, we need not deny that it may be somehow desirable to incorporate into a good formal model of logical properties the informal desideratum that

---

[12] The Continuum Hypothesis is independent of the first-order ZFC (Zermelo-Fraenkel axiomatization of set theory plus the axiom of choice) but semantically decided in its second-order version. Does not this show that the second-order model-theoretic consequence brings in a rather substantive (and controversial) set-theoretical content that cannot be reasonably considered logical? Cf. Blanchette (2001) for a discussion.

[13] Shapiro (1991; 1998; 2005) develops this approach that he calls *logic as model*.

[14] Considerations of economy and simplicity could eventually conspire together to weaken the pull of informal desiderata (intuitions) to the effect that logical truth or consequence has to be necessary, a priori recognizable, or topic-neutral. Thus, what seem to be logically contingent sentences of the type $\exists x(x = x)$, etc. (or arguments having them as conclusions) may eventually be declared as logical truths (valid arguments) of the first-order logic precisely on such "pragmatic" grounds.

they should persist *irrespectively of what may be the case*. The question is how to make it formally precise, that is, how to model it. Now, up to a point at least, the abstract universe of sets is a powerful modelling device that allows us to formally reconstruct both the idea of truth-relevant *cases* (viz. *structures*) and the idea of *truth of a sentence in a case* (viz. a *structure being a model of the sentence*). One can make the idea mentioned above more precise by saying that if a sentence is logically true, then it is true no matter what possible domain it talks of – modelling various possible domains via sets.[15] It is vital here that plenty of domains of various (including infinite) sizes can be represented in the abstract set theory, as this allows us to model the informal idea that logical properties persist no matter how the world could be (that is, no matter what things it may contain or what may be true of them). Just as we need an abstract representation of possible distributions of truth-values w.r.t. atomic sentences, we need an abstract representation of possible distributions of truth-relevant semantic values w.r.t. terms and predicates relative to domains talked about.

Granted, then, the model-theoretic account provides formal reconstructions of informal notions of logical properties, *based* on the mathematical ideology and ontology of the background theory. But, of course, that does not mean that the formally modelled phenomenon is set-theoretic in nature or supervenes on set-theory (Do successful mathematical models of real-world phenomena imply that modelled phenomena are – or somehow supervene on – mathematical phenomena?). Rather, the background set theory (or, possibly, another sufficiently powerful abstract apparatus) may be viewed as an abstract *instrumentarium* of formalized explications of informal interpretational notions of logical properties.[16]

---

[15] Shapiro (1998) and Hanson (1996) both argue for a hybrid interpretational-modal notion of logical consequence as one that the model-theoretic account explicates (though the latter replaces possibilities by set-like domains supposed to exist in the abstract but actual universe of sets). Shapiro (1998; 2005) emphasizes the isomorphism-property of models and suggests that the only differences between models that really matter concern their respective sizes but not what individuals they contain. Shapiro claims that this captures the intuition that logic is in some sense topic-neutral, and, accordingly, that logical truth and consequence are insensitive to identities of objects, being invariant under permutations of the domain.

[16] Except of Shapiro (1998; 2005), see also Chihara (1998), García-Carpintero (1993) and McFarlane (2000).

It is a vexed question how far we can get with the set-theoretic instrumentarium. Already a natural interpretation of the first-order set theory would seem to require the domain (of all sets) that is too large to be a set, hence it is not represented in the model-theory, as model-theoretic structures have set-like domains.[17] And if the world contains more things than can be packed into a set-like collection, then the model-theoretic account misses one crucial possible domain – namely the actual one.[18] Maybe proper classes or something of the sort can be invoked to amend the model-theory in this respect.[19] However, we should never forget that such mathematical models are useful servants only to the extent we understand them well. There are theorists who think that higher-order logics can be approached in a similar model-theoretic spirit, but substantively "higher flights" into set-theory or beyond may be called for if one wants a suitable modelling device for it. And this may certainly give one a pause: are not second-order logical truths and consequences just – as Quine warned us – disguised mathematical (set-theoretical) truths and consequences?

Be that as it may, for the paradigmatic first-order case at least we have an argument in support of the extensional accuracy of MTA, which has no analogue in the second-order or higher-order case. Let us have a closer look at it to see what morals we may draw from it.

---

[17] Viz. Kreisel (1967) and McGee (1992). Reservations about MTA concern the circumstance that domains of models are bound to be sets, which may result in its declaring certain sentences to be true (or false) in all models that are non the less not true (or false) in all interpretations (may be true/false in an interpretation whose domain is too big to be a set). McGee's example is a sentence (involving a new cardinality-quantifier '$\exists^{AI}$' expanding the standard first-order language) to the effect that *there are not absolutely infinitely many (self-identical) things*, where the cardinality in question is of a proper class. Then all set-theoretic interpretations are its models, but it is false under its natural interpretation requiring a proper-class as the domain. See also Blanchette (2000).

[18] Cf. McGee (1992, 279) or Field (2008, 45). Also for Field this means that a sentence *P* (or argument *Ps/C*) may be true (valid) in all set-theoretic structures without being true (truth-preserving) *simpliciter*.

[19] Field (2008, chap. 2) argues that even if we allow more generous models (say, with proper-class domains), the problem arises anew a level higher.

4. Kreisel's squeezing argument

Kreisel (1967) famously argued that the model-theoretic explication of logical validity (as well as of logical truth) introduced above is, in the first-order case, extensionally adequate with respect to a certain informal-intuitive notion of logical validity (or logical truth).[20] Let us consider first-order languages and the notion of logical validity informally characterized as *truth-preservation in every interpretation (structure)*, that is, whatever domain d of individuals we take and whatever extensions (over d) of the right type we let the non-logical vocabulary of the argument to pick out. Kreiselian argument then shows that this informal notion is coextensive with the formally precise notion of *truth-preservation in every set-theoretic interpretation* (*structure*) as it is standardly reconstructed in set-theory.

Kreisel's recipe is remarkably simple.[21] Let I_v be the set determined by the first notion (*intuitive validity*), let S_v be the set determined by the second notion (*set-theoretical validity*), and let P_v be the set determined by the notion of *argument provable in a standard first-order proof-system* (*proof-theoretical validity)*. We start noting that $P_v \subseteq I_v$, as any standard proof-system, is intuitively sound in that it does not prove any argument refutable by some admissible interpretation. That is to say, P_v does not overgenerate w.r.t. I_v, or we would not have a reason to accept the proof-system in the first place. Suppose further that I_v overgenerates w.r.t. S_v. If so, there is an argument that is I_v-valid while having a countable set-theoretic counter-model C in the domain of natural numbers.[22] But this cannot be the case: the argument is not intuitively valid as interpreted in C, hence it is not I_v-valid. We can thus be sure that $I_v \subseteq S_v$. But then we also have

$$P_v \subseteq I_v \subseteq S_v.$$

In the last step we can apply the completeness theorem for first-order logic to obtain $S_v \subseteq P_v$: every set-theoretically valid argument is also proof-theoretically valid. In that case we also have

---

20 Kreisel (1967, 89-93). I shall confine the argument only to logical validity.

21 See Smith (2010) for an admirably clear exposition of squeezing arguments and discussion of their philosophical ramifications.

22 By the downward Löwenheim-Skolem theorem that we saw at work in Quine's argument for extensional adequacy of his substitutional account of logical truth w.r.t. truth in all set-theoretic interpretations (= S_v).

$$P_v \subseteq I_v \subseteq S_v \subseteq P_v$$

which forces the three sets under consideration to coincide in extension:

$$P_v = I_v = S_v.$$

Summing up: Kreisel's reasoning shows that none of the three notions corresponding respectively to the sets P_v, I_v and S_v overgenerates or undergenerates with respect to the others.[23]

Kreisel's recipe has its limits, as for incomplete logics (in particular, second-order logic) we could not carry out the last step of the argument. One issue arising here is what implications this has. One may want to suggest that a genuine logic is to be complete so that a version of Kreiselian argument can be reconstructed for it. But the opponent of this view is likely to retort that a complete (and compact) system such as first-order logic is expressively too weak to formalize categorical theories of paradigmatic mathematical structures (having just one model up to isomorphism; cf. Shapiro 1985; 1991; and Read 1995; 1997), and, connected with this, cannot capture intuitively correct reasoning about such structures (cf. Shapiro 1991 or Read 1995). According to this criterion, second-order logic, albeit not effectively axiomatizable, might be considered superior. In the second round, the opponent of higher-order logic might complain – following Quine (cf. Quine 1986; or Tharp 1975) – that they blur the distinction between logical and mathematical truths. Now, which perspective one deems more plausible and fruitful is going to depend on one's view of what logic is after.

For instance, the fact that MTA for second-order logic declares either CH or non-CH logically true (depending on whether the Continuum Hypothesis holds) may be used to challenge the status of second-order logic as a genuine logic. Or it may be advertised as showing us that, in the general case, MTA does not provide a good model of the informal interpretational notion of logical truth. Or one may want to bite the bullet holding that CH or non-CH is indeed logically true, possibly arguing in a Quinean way – but *pace* Quine himself in this particular case – that there is no clear-cut

[23] Note that Kreisel's squeezing argument does not appeal to the standard soundness theorem relating P_v to S_v.

boundary between logic and mathematics after all (cf. Shapiro 1998, 146, who urges this last strategy).[24]

Another important issue concerns the claim that Kreisel's squeezing argument assures the adequacy of MTA with respect to the intuitive-informal notion of validity. Indeed, the argument appeals to an informal-intuitive notion of "validity" as *truth-preservation under all interpretations in all structures*. But we should be careful here, since the theoretician's "intuitive" notions of validity typically involve a certain sharpening of the initial intuition about consequence that *it is somehow excluded that the premises hold and the conclusion fails to hold* (or that holding of the former establishes holding of the latter).

I propose to see the conceptual situation along the following lines.[25] Kreisel's informal-intuitive notion of validity presents *one* possible sharpening of our initial (imprecise and ambiguous) intuition. Semantically approached, the intuition can be first sharpened via the idea that the conclusion of a valid inference is true in every case in which the premises are true. Second, truth-preservation characteristic of valid arguments is further explained as depending on the semantic behaviour of "formal-logical" elements and on the semantic profile of the premises and conclusion (the pattern of logical terms and the semantic categories of the remaining non-logical terms to which the logical terms are semantically sensitive). This is designed to capture the formal character of logical consequences distinguishing them from those preserving truth due to connections between their descriptive terms.[26]

---

[24] I am indebted to James Edwards for pressing me to be much clearer on this point.

[25] This proposal is indebted to Smith (2010).

[26] It is thus not charitable to charge that they do not capture validity of arguments like "Bob is bachelor; so Bob is unmarried", as Etchemendy (1990; 2008) or Read (1994; 1995) do. Granted, they "undergenerate" w.r.t. the notion of analytical validity (or purely modal notion of validity). But this is just what they wanted! Indeed, we could just as well say that analytical validity overgenerates w.r.t. formal validity. Admittedly, the distinction between analytical and formal validity is relative to how we divide terms into logical and non-logical (or descriptive). If there is – as seems quite likely to me – no principled demarcation, the boundaries of the two classes will be flexible to some extent or other. The debate about the nature of the logical constant is very much alive, but I have no space here to join it. Let it be said that none of various ingenious proposals (including Tarski's 1986 latter attempt to define logical notions as those that are inva-

Precisely this informal explication is captured in Kreisel's "intuitive-informal" notion of *truth-preservation under all interpretations in all structures*, it being understood that (1) structures represent cases and (2) what gets reinterpreted across various structures are non-logical elements. Note, however, that the Kreiselian notion is still "informal", since it is not yet a mathematically defined notion (as opposed, say, to *truth-preservation under all interpretations in all set-theoretic structures*, which is set-theoretically defined as $\mathbf{S}_v$; cf. Smith 2010). However, for the first-order case we have a Kreisel-style argument that this informal notion coincides extensionally with the precise set-theoretical one. For the first-order case, then, we have justified the model-theoretic notion of validity as a good formal explication (or reconstruction) of the Kreiselian notion, which presents an informal explication of the rough intuition that we have started with.

Seen in this light, Kreisel's argument shows that *a* reasonably motivated (and historically important) informal sharpening of the vague idea of validity coincides, in the first-order case, with two formally precise notions – namely model-theoretic validity and provability. That is, the mathematically precise explication of validity in terms of set-theoretical models is not just extensionally adequate with respect to standard proof-systems but also with respect to the interpretationally characterized notion of consequence as truth-preservation under all admissible valuations of non-logical vocabulary that respects the semantics of logical terminology. Accordingly, it can be viewed as a good mathematical model or reconstruction of the last notion.[27]

What Kreisel's argument does not show, though, is that the model-theoretic notion of validity or the informal notion it models is *the* correct one, getting right *the* intuitive notion of validity. On the view I urge, the idea that there is such a notion of validity to be got right is a wild goose chase.[28] If the alleged common notion of validity is the initial intuition

---

riant under all permutations of the domain) has found wider acceptance, and there is a tendency to see the matter of choice of logical constants as more or less pragmatic.

[27] Shapiro (2005) argues, quite plausibly, that the mathematically precise notion of proof-theoretical validity can be seen as a model of an epistemic aspect of consequence understood with Frege as *that which can be derived via a gap-free chain of applications of visibly sound inference rules.*

[28] As I understand them, Smiley (1989), Smith (2010) and Beall – Restall (2006) seem to urge a similar view.

mentioned above, it makes no sense to "get it right" – to give an equivalent preserving all its vagueness and ambiguity. Rather, an explication is in order, whose point is to replace it with something better by way of precision and theoretical fruitfulness. And if one has in mind some notion involving a refinement/sharpening of this intuition, we may point out that several informal explanations of it are possible, the interpretational account being just one among them (though historically prominent). Thus the deductivist or relevantist tradition in thinking about logic propound alternative accounts of the initial intuition about validity that in a valid argument it is somehow excluded that the premises hold and the conclusion fails to hold. The question as to which of these notions is *the* correct one is out of place, though we may compare them and weigh their merits in light of their theoretical fruitfulness, comparative clarity of ideology or ontological costs.[29]

5. The alleged devastating objection

The foregoing discussion will not persuade everybody that quantificational account in the model-theoretic style is a good thing. Let me finally turn to what can be considered the most principal objection to quantificational account in whatever form. An early version was voiced by Kneale in the early 1960s:

> Just as according to [Bolzano's] definitions a proposition can be analytically true by accident, so too one proposition may follow from another by accident, that is to say in such a way that the truth of the universal proposition about the results can be known only by an examination of the individual results.
> ... a proposition cannot properly be said to be derivable from a set of premises unless it is possible to establish that if the premises are true the proposition is also true without first establishing whether or not the premises and the proposition are true. (Kneale 1961, 94)

[29] Moreover, there is nothing in the initial intuition *per se* that compels us to emphasize the formal aspect and the classical first-order forms in particular (hand in hand with its standard selection of logical terms). So there is a room for accounts of validity explicating the vague intuition so as to be usable for "non-classical" logics, which develop the logic of specific modal, epistemic, deontic (etc.) notions.

Kneale's worry is that if we explain the nature of logical properties in a purely quantificational style (substitutionally or interpretationally), we cannot do justice to the intuition that recognition of an argument's validity is to be independent of knowledge of the truth-values of its component sentences.

An elaborated version of this objection can be found in Etchemendy's sustained argument against the standard model-theoretic account (see especially Etchemendy 2008, 265-271). On his view, quantificational accounts in general and the standard model-theoretic account in particular introduce as a defining characteristic of validity something that is only a symptom (though a reliable one) of it. Let us call this symptomatic characteristic, which only masquerades as the true cause of validity, *the quantificational condition*:

Q: to belong to a class of equiform arguments containing only truth-preserving arguments.

It is plain wrong, Etchemendy claims, to say that *A* is valid just *because* it meets *Q*, that is, just because it has only truth-preserving variants in the same form. As Kneale pointed out, this does not make *A* conceptually different from arguments that by sheer coincidence have no equiform counterexample, whose formally truth-preserving character is thus accidental and as such would have to be ascertained empirically – instance by instance.

Already this, Etchemendy submits, is a *reductio* of all quantificational accounts. Due to their faulty conceptual analysis that mistakes symptoms for a cause, quantificational accounts cannot but miss the following desideratum

I. whatever validity is, it must be an intrinsic feature of *A* whose possession by *A* is recognizable without knowing the actual truth-values of *A*'s components (or of any other argument),

a special corollary of which is that

II. whatever validity is, it must be an intrinsic feature of *A* that provides a guarantee that *A*'s conclusion is true given that *A*'s premises are jointly true, which does not depend on one's knowledge of the truth-value of *A*'s conclusion.

If one is after a plausible conceptual analysis of logical validity, one is to capture in the *analysans* a characteristic that meets at least those two desiderata. Or so Etchemendy seems to suggest. However, quantificational accounts relying on *Q* (or something of the sort) do not supply any such desirable characteristic. It seems that one cannot recognize that *A* meets the condition *Q*, without knowing of each given argument *A** in the same form that it is truth-preserving. Moreover, even if one checked all the other arguments in the same form and found them truth-preserving, this would not give him/her independent assurance that *A* is truth-preserving, not even if one already knew its premises to be true. For all one knows at this juncture is: either *A* has a false conclusion and is invalid or *A* has a true conclusion. So only if one already knows that *A* has a true conclusion, can one be sure that *A* is valid (provided one also knows that all other arguments in the same form preserve truth)! In which case, however, validity so defined is of no use at all to justify one in accepting *A*'s conclusion on the strength of *A*' premises. Etchemendy concludes:

> It is clear that Tarski's definition tries to reduce a 'cause' – the logical consequence relation – to its 'symptoms,' the truth preservation that the consequence relation guarantees. And it is equally clear that this guarantee of truth preservation is the essential feature of logical consequence, the feature that makes it possible to infer the conclusion of a valid argument from its premises. In short, the reductive analysis omits the single most important characteristic of the consequence relation. (Etchemendy 2008, 271).

6. The alleged devastating objection rebutted

For a start, we should note that *Q* is not a sufficient but only a necessary condition of logical consequence in Tarski's interpretational or the standard model-theoretic account. But Etchemendy could still argue that a version of his argument goes through even for quantificational conditions spelled out in terms of interpretations. Indeed, there is a reason to think that if his argumentation works at all, it applies, *mutatis mutandis*, to any account spelled out in terms of *truth-preservation in all cases*, no matter how *cases* are construed:

> *A* follows from Γ just if *A* is true in every case *C* in which every $Ai \in \Gamma$ is true.

Already this, I think, may warn us that something is wrong with the argument. For note that much the same reasoning would apply to the "representational" slogan favoured by Etchemendy:[30]

> *A* follows from Γ just if for every possible configuration *w* of the world, *A* is true in *w*, if every *Ai* ∈ Γ is true in *w*.

Whatever *cases* may be, one could argue in Etchemendy's style that *truth-preservation (truth) in all cases* is merely a symptom of what in the last instance brings it about (of a "true cause"). For surely it is no less absurd to suppose that one has to check all logically possible ways the world could be in order to find assurance that truth cannot but be preserved from Γ to *A*. Even if, *per impossible*, we checked all possible but non-actual ways the world may be and found out that they do not disqualify the inference from Γ to *A*, and even if, in addition, we knew that all the premises are true, we would still face the following unpalatable option: either *A* is non-truth-preserving (invalid) in the actual world or *A* has a true conclusion in this world. In order to decide the question whether the inference is valid, we would have to know the actual truth-value of *A*'s conclusion.

One thus suspects that the argument shows at most that quantificational slogans are *just* slogans and cannot therefore provide the whole story. Indeed, at least in interpretational approaches there is arguably more to logi-

[30] Etchemendy (2008, 285-295) compares the interpretational approach to model-theoretic semantics (developed by Tarski *et al.* in the 1950s) with the representational approach to model-theoretic semantics. Roughly speaking, while the first fixes the world and lets interpretations of the non-logical vocabulary vary (together with the domain of quantifiers), the second fixes meanings of all words and lets the world vary (but allowing words to pick out different extensions in different logically possible configurations of the world). To be fair to him, he does not think that the slogan spelled out in terms of logically possible worlds provides a conceptual analysis of logical validity. In his view, the notion of "logically possible configuration of the world" already presupposes understanding of logical properties, since representational models must be *consistent*, mutually *independent* and *jointly complete* in determining the whole logical space of possibilities. If I understand him, Etchemendy thinks that logically possible configurations of the world invoked by a representational model-theory are not irreducibly "modal" in some metaphysical sense, but reflect the specifics and requirements of a proper semantic analysis of a given domain of discourse (reasoning), which focuses on the semantics of certain terms (but without assuming any fixed-privileged set of "logical" terms), while treating the remaining terms "schematically" (what general semantic features they contribute to the semantic structure involving the first terms).

cal properties than the quantificational slogan reveals. By way of conclusion, I am going to point out that this is something that should be rather obvious in the case of the standard model-theoretic account. In my view, it makes Etchemendy's argument off the mark.

The minor point already mentioned is that MTA does not aim at a conceptual analysis but at a mathematical explication (model) of informal logical notions.[31] But, more importantly, the quantificational slogan is not all there is to the model-theoretic reconstructions of logical properties. It is but a convenient short hand for something more complex, which presupposes the full-blooded model-theoretic explanation of satisfaction/truth in a set-theoretic structure with all its tedious details. It is these details, of course, that flesh out the quantificational slogan spelled out in terms of truth-preservation in all (set-like) structures. The recursive story fixes the semantics of logical operators and shows how the truth-value of every sentence (in a given structure) is determined based on the truth-relevant features of its components to which the fixed semantic features of logical operators are sensitive. Importantly, this story also tells us that there are sentences or arguments that possess certain properties independently of how their non-logical vocabulary is interpreted – no matter what specific values of fitting types from what particular domain they pick out. So to check that they possess such properties we need to have a general semantic knowledge of what the general recursive story states, but no specific knowledge of what specific values non-logical elements have (that is, to know the fixed semantic roles of specifically logical operators and the principled ways in which the non-logical vocabulary contributes to fixing truth-values of sentences whose semantic structure is determined by a certain schematic pattern involving the logical vocabulary).[32]

Up to a point, this answers Etchemendy's objection that the interpretational account (formalizable model-theoretically) completely misses the epistemic aspect of validity (he conceives of it as a kind of *a priori* knowability that the supporting relation between the premises and conclusion obtains). But in the next step one may question his desiderata I-II, as begging the

---

[31] Apparently, I disagree with Etchemendy on this point too, as he thinks that interpretational accounts are failed attempts to provide accurate conceptual analysis of logical properties. Cf. Etchemendy (2008, 294).

[32] A well executed attempt to justify the model-theoretic account along these lines is García-Carpintero (1993).

question against semantic-interpretational notions of validity (logical truth). On such accounts, the "semantic" consequence-relation, unlike the "syntactic" proof-relation, need not be in general effectively recognizable – not even a priori. If an argument belongs to a decidable logical system (viz. propositional logic), everything is just fine. When it belongs to a complete system whose set of theorems (logical truths) is recursively enumerable but not decidable (such as first-order logic), we have at least a positive test: if the argument *Ps*/*C* is valid, there is a proof in the system of *C* from *Ps*. However, when it belongs to a system with an incomplete proof-procedure (viz. second-order logic), there is not even a positive test. Now the proponents of the model-theoretic approach in particular tend to see this as its virtue helping us to sort things out, rendering the deductive and the semantic side of logic formally tractable and making room for fruitful meta-theoretic comparisons between them.[33]

Maybe Etchemendy does not have in mind proof-theoretic criteria when he speaks of the intrinsic power of valid arguments to justify conclusions solely on the strength of accepting their premises. Maybe he thinks that a properly "semantically" construed account of logical properties – along the representationalist lines – must show that valid arguments have this epistemic (justificatory) feature. If so, he just does not make it clear how such a story is supposed to go (and I suspect that its essentials, if spelled out, would not significantly differ from the account given two paragraphs back). Hence it is far from clear why a semantically valid argument should always be (*a priori*) recognizable as such.

All in all, seeing how the recursive semantic story fleshes out interpretational accounts and that the semantically construed consequence-relation, unlike the effective proof-relation, may not be (always) recognizable as such, the proponents of interpretational approaches (including its model-theoretic formal explication) need not be paralysed by Etchemendy's allegedly devastating objection.

7. Conclusion

My aim in this study was to show that MTA, properly understood, marks a culmination point in the quantificational tradition; and that, up to

[33] Here I am much indebted to the comments of James Edwards.

a point, it can be partially vindicated against various objections challenging its adequacy. The strategy was to show that, up to a point at least, MTA is a good formal explication of a specific informal semantic account, according to which logically valid arguments preserve truth in virtue of their logico-semantic structure and irrespectively of particular semantic values of the non-logical vocabulary. This is a modest achievement. For one thing, I make no pretentious claims to the effect that MTA is the best account of *consequence proper*. On the pluralistic approach urged here, this claim does not even make good sense, since there is no such a thing as *the* intuitive or pretheoretic notion of consequence to be captured by the conceptually adequate definition. For another thing, there are foundational questions about consequence that I have not touched. For instance, if logic indeed aims to provide good models of correct reasoning, one pertinent issue is whether mathematical practice in particular and reasoning practices in general are not better reflected in the *deductivist* or *inferentialist* models, which give pride of place to rules of inference and their chaining. But, understandably, this big issue – and related questions – could not be addressed in the limited stace of this study.

Acknowledgements

I am grateful to Karel Procházka, Jaroslav Peregrin and James Edwards, in particular, for corrections and instructive comments on earlier drafts of this study.

References

BEALL, J.C. – RESTALL, G. (2006): *Logical Pluralism*. Oxford: Clarendon Press.

BLANCHETTE, P. (2000): Models and Modality. *Synthese* 124, 45-72.

BLANCHETTE, P. (2001): Logical Consequence. In: Goble. L. (ed.): *The Blackwell Guide to Philosophical Logic*. Blackwell, 115-135.

CARNAP, R. (1937): *Logical Syntax of Language*. London: Kegan Paul.

CHIHARA, C. (1998): Tarski's Thesis and the Ontology of Mathematics. In: Schirn, M. (ed.): *The Philosophy of Mathematics Today*. Oxford: Oxford University Press, 157-172.

ENDERTON, H. (2001): *A Mathematical Introduction to Logic*. 2nd edn. San Diego: Academic Press.

ETCHEMENDY, J. (1990): *The Concept of Logical Consequence*. Cambridge (Mass.): Harvard University Press.

ETCHEMENDY, J. (2008): Reflections on Consequence. In: Patterson. D. (ed.): *New Essays on Tarski and Philosophy*. Oxford: Oxford University Press, 263-299.

FIELD, H. (2008): *Saving Truth from Paradox*. Oxford: Oxford University Press.

GARCÍA-CARPINTERO, M. (1993): The Grounds for the Model-Theoretic Account of the Logical Properties. *Notre Dame Journal of Formal Logic* 34, 107-131.

HANSON, W.H. (1997): The Concept of Logical Consequence. *Philosophical Review* 106, No. 3, 365-409.

HODGES, W. (1997): *A Shorter Model Theory*. Cambridge: Cambridge University Press.

KNEALE, W. (1961): Universality and Necessity. *The British Journal for the Philosophy of Science* 12, 89-102.

KOREŇ, L. (2014): Quantificational Accounts and the Problem of Logical Consequence II: In the Footsteps of Bolzano. *Organon F* 21, No. 3, 303-326.

KREISEL, G. (1967): Informal Rigour and Completeness Proofs. In: Lakatos, I. (ed.): *Problems in the Philosophy of Mathematics*. Amsterdam: North-Holland, 138-171.

MACFARLANE, J. (2000): What is Modeled by Truth in all Models? Preprint available on-line at: johnmacfarlane.net/whatismodeled.pdf.

MCGEE, V. (1992): Two Problems with Tarski's Theory of Consequence. *Proceedings of the Aristotelian Society* (New Series) 92, 273-292.

PRIEST, G. (1995): Etchemendy and Logical Consequence. *Canadian Journal of Philosophy* 25, No. 2, 283-292.

PRIEST, G. (1999): Validity. *European Review of Philosophy* 4, 183-205.

QUINE, W.V.O. (1986): *Philosophy of Logic*. 2nd ed. Cambridge (Mass.): Harvard University Press.

READ, S. (1994): Formal and Material Consequence. *Journal of Philosophical Logic* 23, 247-265.

READ, S. (1995): *Thinking about Logic*. Oxford: Oxford University Press.

READ, S. (1997): Completeness and Categoricity: Frege, Gödel and Model Theory. *History and Philosophy of Logic* 18, No. 2, 79-93.

RUSSELL, B. (1918/1919): The Philosophy of Logical Atomism. *Monist* 28, 495-527; *Monist* 29, 32-63, 190-222, 345-380. Reprinted in Russell, B. (1956): *Logic and Knowledge*. London: Allen and Unwin, 177-281.

RUSSELL, B. (1919): *Introduction to Mathematical Philosophy*. New York: Macmillan.

SHAPIRO, S. (1985): Second-Order Languages and Mathematical Practice. *Journal of Symbolic Logic* 50, 714-742.

SHAPIRO, S. (1991): *Foundations without Foundationalism: A Case for Second-Order Logic*. Oxford: Clarendon Press.

SHAPIRO, S. (1998): Logical Consequence: Models and Modality. In: Schirn, M. (ed.): *The Philosophy of Mathematics Today*. Oxford: Oxford University Press, 131-156.

SHAPIRO, S. (2002): The Status of Logic. In: Boghossian, P. – Peacocke, Ch. (eds.): *New Essays on A Priori*. Oxford: Oxford University Press, 333-356.

SHAPIRO, S. (2005): Logical Consequence, Proof Theory, and Model Theory. In: Shapiro, S. (ed.): *The Oxford Handbook of Philosophy of Mathematics and Logic*. Oxford: Oxford University Press, 651-670.

SMILEY, T. (1989): Consequence, Conceptions of. In: Craig, W. (ed.): *Routledge Encyclopedia of Philosophy*. London: Routledge, 599-603.

SMITH, P. (2010): Squeezing Arguments. *Analysis* 71, No. 1, 22-30.

TARSKI, A. (1936): On the Concept of Logical Consequence. In: Tarski, A. (1983): *Logic, Semantics, Metamathematics*. 2nd ed. Corcoran, J. (ed.), Indianapolis: Hackett, 409-420.

TARSKI, A. (1986): What are Logical Notions. *History and Philosophy of Logic* 7, 143-154.

THARP, L. (1975): Which Logic is the Right Logic? *Synthese* 31, No. 1, 1-21.