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Quantificational Accounts of Logical Consequence II: In the Footsteps of Bolzano[1]

LADISLAV KOREŇ

Department of Philosophy and Social Sciences. Faculty of Arts. University of Hradec Králové
Náměstí Svobody 331. 500 03 Hradec Králové. Czech Republic
ladislav.koren@uhk.cz

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ABSTRACT: Quantificational accounts of logical consequence account for it in terms of truth-preservation in all cases – be it admissible substitutional variants or interpretations with respect to non-logical terms. In this second of my three connected studies devoted to the quantificational tradition I set out to reconstruct the seminal contributions of Russell, Carnap, Tarski and Quine and evaluate them vis-à-vis some of the most pressing objections. This study also prepares the ground for my discussion of the standard model-theoretic account of consequence to be found in the concluding study.

KEYWORDS: Logical consequence – quantificational accounts – substitutions – interpretations – persistence.

1. Introduction

The idea that we could fruitfully explicate the notion of *C logically following from P* as truth-preservation under all admissible variations with respect to all non-logical elements in $P \cup C$ has proved extremely influential in the western logico-semantic tradition.[2] Attempts to explain consequence following this recipe are sometimes called *quantificational accounts*. My first

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[2] *P* and *C* represent the premise-set and the conclusion-set respectively.

study (see Koreň 2014) devoted to the quantificational tradition revolved around three contentions. First, quantificational accounts give pride of place to the formal dimension of consequence. Second, what marks them out from other approaches that likewise emphasise the formal aspect of consequence is the fact that, one way or another, they propose to explain away, or reduce, the modal element that resurfaces in informal glosses on consequence and related notions. Third, full-blooded accounts fitting this description seem to have emerged only with Bolzano. Bolzano, unlike his distinguished predecessors, was prepared to take the bold step of accounting for consequence and related logical ideas solely by generalizing over appropriate cases (admissible substitutional variants) without having to appeal to irreducibly modal notions.

In this follow-up study I am about to explore the quantificational tradition following in the footsteps of Bolzano.[3] I start with Russell's account of modal and logical notions as representing specific properties of propositional functions, not least because it provoked a principal objection due to Wittgenstein, the gist of which seems to pose a *prima facie* challenge also to modern substitutional and interpretational accounts devised for languages of mathematical logic. Thus I shall reconstruct two important *objections from overgeneration* in connection with the quantificational accounts of Carnap (cf. Carnap 1937), Tarski (cf. Tarski 1936) and Quine (cf. Quine 1970/1986) and their ramifications. I then go on to spell out what I consider the main residual worries, suggesting that they gesture towards the standard model-theoretic approach as a superior quantificational account that promises to assuage them. Whether this conjecture can be vindicated vis-à-vis a battery of heawyweight objections levelled by the modern critics of the model-theoretic account is a delicate issue,[4] whose treatment is left for the concluding part of my explorations.

2. In the footsteps of Bolzano

2.1. Russell on modal and logical notions

Bolzano's substitutional account reduces logical truth of a sentence *A* to the universal truth of a sort: truth under all admissible variations with re-

3 I summarized some of them in the concluding section of Koreň (2014).

4 Etchemendy (1990) has been the most influential critical voice.

spect to its non-logical elements. Bolzano thought that the same holds for the relation of *C logically following from P*.[5] He claimed that Aristotle's turn of phrase "results of necessity" occurring in his classic account of *deduction* can only be understood in terms of the *whenever*-connection between P and C.[6] And this connection he explained as truth-preservation under all admissible variations w.r.t. the set V containing all the non-logical elements occurring in $P \cup C$.[7] Whether or not he was right about Aristotle's intentions is debatable, to say the least. What is not debatable is the fact that Bolzano made a very intriguing proposal that has proved attractive to many philosophers since then.[8]

Interesting affinities can be found in Russell's explications of the modal notions of *possibility*, *impossibility* and *necessity*. In his widely read lectures on the philosophy of logical atomism Russell (1918/1919) argues that such notions do not apply to propositions, but to "propositional functions". The reason is that, once we read "*A* is possible" as "*A* is sometimes true" or "*A* holds in some cases", this indicates that we can make sense of *A*'s having cases or instances. Yet only something with undetermined elements can have cases or instances. Such things are propositional functions with variable elements whose values for various definite arguments replacing the variables are various determinate propositions. Or so Russell argued.

Russell then goes on to say that a propositional function $\Psi(x_1,\ldots,x_n)$ is *possible* if it holds in *at least one* propositional instance of it, that is, if it yields a true proposition for at least one admissible substitution for all its free variables. Now this amounts to reducing possibility to the truth of an

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5 Bolzano (1837/1972) employed the term *deducibility* for the generic relative consequence-relation, that is, C following from P with respect to a set V of variable elements (not necessarily all and only the non-logical elements) occurring in $P \cup C$. Logical consequence requires the set V to contain all and only the non-logical elements occurring in $P \cup C$. See Bolzano (1837/1972, §29). The same applies, *mutatis mutandis*, to his notion of *logical analyticity* – cf. Bolzano (1837/1972, §148).

6 Recall the *locus classicus*: "[…] deduction is a speech in which, some things having been supposed, something other than what has been supposed results of necessity from their being so" (Aristotle 1964, 24b18-22). Bolzano's gloss is as follows: "[…] the 'follows of necessity' can hardly be interpreted in any other way than this: that the conclusion becomes true whenever the premises are true" (Bolzano 1837/1972, §155, §§219-220).

7 Henceforth, I use "w.r.t." to abbreviate "with respect to".

8 Łukasiewicz (1957) is one influential commentator who agreed with Bolzano that Aristotle implicitly subscribed to this approach.

existential proposition $\exists x_1,\ldots,x_n\Psi(x_1,\ldots,x_n)$. In a similar spirit, $\Psi(x_1,\ldots,x_n)$ is said to be *necessary* if it holds in *every* propositional instance, that is, if a true proposition results for every admissible substitution for all its variables. This, again, comes to reducing necessity to the truth of a universally quantified proposition of the type $\forall x_1,\ldots,x_n\Psi(x_1,\ldots,x_n)$.[9]

This is not yet Russell's account of specifically *logical necessity*. Russell thought that genuine logical truths are law-like propositions concerned with the real world, with which the process of abstraction and generalization reached as it were its "utmost limit". Thus logic, he famously said, "is concerned with the real world just as truly as zoology, though with its more abstract and general features" (Russell 1919, 169). It follows, according to Russell, that genuine logical truths are fully generalized propositions composed solely of logical elements together with variables of appropriate logical types, none of which refers to any specific contents of the world. And this complete abstraction from the specific contents of the world is what renders logical truths *formal*, hence topic-neutral (but we shall see shortly that there is another sense that Russell attaches to the notion of formality – i.e. the Wittgensteinian idea of truth by virtue of a logico-syntactic make-up alone, hence irrespectively of possible ways the world could be – that does not coincide with the former sense).

To clarify what this amounts to we should note that for Russell neither the proposition

(1) Oscar is a philosopher or Oscar is not a philosopher

nor its first-order universal closure

(2) ∀x(x is a philosopher or x is not a philosopher)

is strictly speaking logically true (necessary), since neither is purely formal in that both involve reference to a specific subject-matter. What, according to Russell, would qualify as a genuine logical truth (law) is the second-order universal closure with respect to all the topic-sensitive elements:

(3) ∀X∀x(x is X or x is not X).

In a sense, however, we can say that (1) or (2) are logically true (necessary), though in a derived way, being specific instances of (hence deducible

[9] In an analogous manner, impossibility is defined as non-existence of a verifying propositional instance of a propositional function. See Russell (1919, 162).

from) the completely general logical law (3). In this way, then, logical truth can be reduced to the truth of a completely general proposition.

This, I take it, should remind us of Bolzano's substitutional account in that the universal closure of all (including, possibly, higher-order) variables has a similar effect as the talk about the truth of *all* admissible instances of a propositional form (e.g. *x is X or x is not X*) irrespectively of what non-logical elements of fitting types uniformly replace variable elements (cf. Corcoran 1973 and Sagüillo 2002).

Incidentally, Bolzano's and Russell's accounts of logical consequence are close too. Using the familiar conditional-manoeuvre, Russell reduces the relation of logical consequence between the (finite) premise-set $\{A_1, \ldots, A_n\}$ and the conclusion B to the truth of a universal closure of the conditional

If A_1^* and … and A_n^*, then B^*,

where the starred letters stand for the corresponding propositional functions that do not contain any non-logical elements but only logical constants together with variables of appropriate types. For instance, given that we treat '=' as a fixed logical constant, to assert

"$\neg(17 = 6)$" follows logically from "17 is prime and 6 is not prime"[10]

is a way of asserting something general about the logical propositional function

If $X(x)$ and $\neg X(y)$, then $\neg(x = y)$.

In fact, it is something that we could express by its second-order universal closure

$\forall X \forall x \forall y$(if $X(x)$ and $\neg X(y)$, then $\neg(x = y)$).

With this higher-order truth the process of logical generalization has finally reached its utmost limit.

2.2. Wittgenstein's principal challenge

According to Russell, then, logical consequence reduces to logical truth. And the latter reduces itself to a fully general truth of a sort – truth irrespectively of the specific referents of non-logical terminology.

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[10] Russell's preferred idiom was: *A formally implies B*.

Wittgenstein famously complained that this Russellian conception of logical truth – as a fully general truth – insufficiently distinguishes logical truths from mere generalities that could be only accidental:

> The mark of a logical proposition is *not* general validity... An ungeneralized proposition can be tautological just as well as a generalized one. (Wittgenstein 1921, § 6.1231)
> Logical general validity, we could call essential as opposed to accidental general validity, e.g. of the proposition "all men are mortal". (Wittgenstein 1921, § 6.1232)

What Wittgenstein claims here is that generality is neither necessary nor sufficient for logical truth. As regards the first point, he would contend that the claim made by (1) is tautological if anything is. He thus attacks Russell's view, according to which the status of (1) as a logical necessity is at best derivative: it can be called logically true, being an instance of the completely general law expressed by the claim (3). But, by Wittgenstein's lights, for a proposition to qualify as a logical truth it must be a vacuous tautology holding independently of factual matters, accordingly enjoying *a priori* status, as any recourse to empirical evidence is out of question (cf. Wittgenstein 1921, §6.1, §6.11). Now, (1) is such a tautology, as its elementary truth-functional character testifies.

As for Wittgenstein's second point, what he had in mind is that completely generalized truths may well express only something very general about reality. Precisely because of that, however, the possibility is always open that they hold only accidentally in that the reality may just happen to possess this general structure (or feature) rather than a different one:

> Our fundamental principle is that whenever a question can be decided by logic at all it must be possible to decide it without further ado. (And if we get into a position where we have to look at the world for an answer to such a problem that shows that we are on a completely wrong track.) (Wittgenstein 1921, §5.551)

Thus, once we assign logical propositions a subject-matter – be it completely general or, perhaps, about peculiar logical objects of a sort – we have failed to separate them principally from empirical propositions, and, in particular, from *a posteriori* generalizations:

> All theories that make a proposition of logic appear to have content are false. [...] On this theory it seems to be anything but obvious, just as,

> for instance, the proposition, 'All roses are either yellow or red', would not sound obvious even if it were true. Indeed, the logical proposition acquires all the characteristics of a proposition of natural science and this is the sure sign that it has been construed wrongly. (Wittgenstein 1921, §6.111)

Their truth would thus depend on *how things are* so that to recognize them as true we would presumably have to check the facts to confirm whether it is this (general) way rather than any other (general) way. Yet this is completely misguided if, as Wittgenstein has it, logical propositions are distinguished from factual-empirical propositions precisely in that

> […] one can recognize that they are true from the symbol alone, and this fact contains in itself the whole philosophy of logic. (Wittgenstein 1921, §6.113)
> That is to say, if we know the logical syntax of any sign language, then all the propositions of logic are already given. (Wittgenstein 1921, §6.124)

This formal dimension has a semantic counterpart. Holding (or not) irrespectively of how things are, logical propositions do not describe reality but determine the very structure of the whole logical space of combinatorial possibilities. There is therefore no genuine reference to the factual-empirical, hence no genuine subject-matter – not even a completely general one.

2.3. Russell's way of addressing Wittgenstein's challenge

Returning now to Russell, he was well aware of Wittgenstein's challenge. He tried to fix the problem – apparently influenced by Wittgenstein – by contending that logical truths are to be not just fully general but also *tautological* in the specific sense of being true in virtue of their logico-syntactic make-up, hence irrespectively of the possible ways the world could be. As he also put it, they are to be true *in virtue of form* (cf. Russell 1919, 197). Of course, this manoeuvre ignores Wittgenstein's first point that complete generality is not necessary for logical truth, given that propositions such as (1) are logically true. Yet it appears to make at least some progress with regard to the second objection that complete generality does not guarantee logical truth. Russell agrees it does not. He denies, for instance, that sentences like "There is at least one thing" are truths of pure logic, though they may be expressed in purely logical words.[11] What he

[11] E.g. formalized as $\exists x(x = x)$, where identity is treated as a logical symbol.

tentatively suggests, I submit, is that *generality* + *formality* could provide such a guarantee. His idea seems to be, first, that truly logical propositions are truly general (abstract, topic-neutral) laws. Second, since such logical laws are the (unempirical) source of validity of their (less general) instances, the later inherit from the former specifically logical necessity.

Unfortunately, Russell's account is rather obscure in this crucial respect. Thus, having said that the form cannot be one of the constituents of the proposition whose form it is – otherwise, what would hold the form and the other constituents together? – Russell tentatively suggests that it might be a subject matter of another logical proposition so that

> [...] it is possible that logical propositions might be interpreted as being about forms. (Russell 1918/1919, 75)

Fully general logical propositions, recall, are not about specific things, properties or relations, but, presumably, they are not completely devoid of subject matter either. Russell sometimes talks as if the formality of a logical proposition consisted precisely in the fact that its subject matter is a logical form:

> [...] another way of stating the same thing is to say that logic (or mathematics) is concerned only with forms, and is concerned with them only in the way of stating that they are always or sometimes true – with all the permutations of "always" and "sometimes" that may occur. (Russell 1919, 199-200)

Yet he felt rather insecure about this – and not without reason. On the one hand, the passage confirms the analogy with Bolzano: a logical truth such as (3) says, in effect, that the form (F) *x is X or x is not X* holds in all instances, for all admissible values of the variables "X" and "x". On the other hand, how does the fact that (3) is about (F) show that it itself is true in virtue of form? Indeed, in virtue of what form? In virtue of (F), which is supposed to feature as a constituent in its subject matter? That seems confused, as, intuitively, (F) is not the form of (3), but of its instances such as (1). And Russell cannot say that (3) also displays (F), because he has maintained that no form of a proposition can be a constituent of its subject matter. So, particular instances of (3) could perhaps be said to be true in virtue of the form (F), but they are not fully general propositions; and while (3) is fully general, it is not true in virtue of (F).

Keeping the spirit of Russell's approach, we could tentatively suggest the following. The fully general proposition (3) is *sui generis* in that it displays the form such that the class of propositions with the same form is the class containing nothing but (3). In this way, we could maintain that (3) is true in virtue of its form, since the only proposition of this form is true. Moreover, (3) is fully general. So (3) can be deemed logically true, as it is both completely general and true in virtue of form.

Unfortunately, even this amended proposal fails to address Wittgenstein's challenge, because it does no separate purely logical laws from contingently true generalities expressed in a purely logical idiom. As Russell pointed out himself, the proposition such as "There is at least one thing identical with itself" – and, in general, cardinality statements – can be translated into the purely logical idiom à la *Principia Mathematica* (cf. Russell 1919, 203; see also footnote on the same page). Yet it seems to state something substantive and contingent about the way the world is, which need not hold in different ways the world could be. Accordingly, its truth cannot be recognized on *a priori* grounds, which epistemic quality Russell clearly expects genuine logical truths to possess.

I noted that Russell was aware of the difficulty and proposed to attack it with glosses inspired by Wittgenstein's conception of tautology. Without any real progress, as far as I can judge. Indeed, he confessed to be unable to explain in a satisfactory way his Wittgenstein-inspired notion of tautology as capturing one essential mark of (logical) *analyticity*, whose other side is *a priority*.[12]

In what follows I am about to show that modern quantificational accounts propose a somewhat different approach: instead of seeking a unified account of genuinely logical laws ultimately grounding all logical necessities, they aim to formulate a meta-theoretic account of logical consequence and truth in terms of truth-preservation under all admissible variations of a sort. But we will also see that they inherit some of the central difficulties that confronted Russell's approach.

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[12] See Russell (1919, 205). Another serious trouble is that if the idea of reduction of all logical truths (necessities) to general logical laws is cashed out as their deductive encapsulation in the latter, it founders on Gödel's first incompleteness theorem (Gödel 1931), provided that deduction from logical laws is understood in the standard proof-theoretic sense.

3. From substitutions to interpretations

3.1. Carnap: substitutional account for formalized languages

Another important contributor to the quantificational tradition was Rudolf Carnap. In his monumental *Logical Syntax* he provided substitutional accounts of logical notions, though he proceeded in a reversed order. He proposes first to define logical consequence this way:[13]

> B is a logical consequence of $A_1,\ldots,A_n$ iff B is a consequence of $A_1,\ldots,A_n$ and either (1) B and $A_1,\ldots,A_n$ are all logical sentences containing only logical expressions, or (2) B^* is a consequence of $A_1^*,\ldots,A_n^*$, where B^* and $A_1^*,\ldots,A_n^*$ are any admissible substitution-variants of B and $A_1,\ldots,A_n$ respectively, obtainable by uniformly replacing all descriptive expressions occurring in the latter sentences with other descriptive signs of the same logical type.[14]

This definition itself presupposes the definition of the generic consequence-relation in terms of what can be obtained from a premise-set by means of repeated applications of certain transformation rules.[15] With this in hand, Carnap accounts for analyticity (logico-analytical truth) as follows:

> A is analytic iff A is a consequence of the empty premise-set and either (a) A contains only logical vocabulary, or (b) A is such that every sentence obtainable from it by uniformly replacing all its descriptive signs with other descriptive signs of the same logical type is true (cf. Carnap 1937, 181; see also Coffa 1991).

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[13] For details see Coffa (1991), Creath (1998), Procházka (2006) and several essays in Wagner (2009), especially de Rouilhan (2009).

[14] I have simplified the definition, leaving out of account the case when the descriptive signs in premises or conclusion are defined. In that case, the conclusion logically follows from the premise-set iff the logical consequence relation (as defined above) holds between their variants in which all non-primitive terms are everywhere replaced by their *definiens*.

[15] Note, though, that the consequence-relation is infinitary in character: basically, it is explained in terms of what can be obtained from a premise-set by means of repeated applications of possibly infinitary transformation rules such as the so-called omega-rule allowing us to infer the universal closure ∀xPx from the infinite premise-set that includes P(n), for each given natural number n. Introduction of such infinitary ("indefinite") deductive means was an important element in Carnap's original way of accommodating Gödel's incompleteness results within his generously syntactic project, fully acknowledging their limitative force with regard to finitary ("definite") means of proof.

This is one of three definitions of logical properties given in Carnap's *opus magnum*.[16] It can be found in the important part on the general syntax, in which Carnap attempts to reconstruct all the key notions of logical consequence, analyticity, contradictoriness and determinacy – including, importantly, the partition of all terms into logical and descriptive – starting with the definition of the consequence-relation. The other two definitions were given for the so-called Language I and Language II respectively. For Language I, Carnap's recipe was much like the general one provided above: viz. analyticity was defined as a limiting case of logical consequence from the empty premise-set. However, for the much stronger LII (of the type-theoretic sort), he defined logical consequence as a limit-case of L-contradictoriness: B follows logically from $A_1, \ldots, A_n$ iff $\{A_1, \ldots, A_n\} \cup \{\neg B\}$ is contradictory. Importantly, Carnap's account of analyticity and contradictoriness made use of the method of valuation and was remarkably close to Tarski's celebrated procedure of defining truth via a recursive definition of satisfaction of open sentences by sequences of objects (of fitting logical types).[17] Basically, the idea was that analyticity of complex (including quantified) sentences can be systematically reduced, in a step-by-step manner, to analyticity of atomic sentences.

In Carnap's view, this procedure amounted to the definition of *logical truth* for sentences of Language II articulated in purely logical (logico-mathematical) expressions. A problem with this way of defining logical truth is that it does not neatly extend to sentences containing descriptive constants. Yet it was arguably Carnap's ambition to provide a general method of defining analyticity (and related notions of contradictoriness and consequence) also for descriptive languages of science (e.g. for physicalistic extensions of Language II). Carnap's proposal here was that in the specific case of a descriptive sentence A we san say that:

A is analytical (logically true) iff A^* is analytical, where A^* is a sentence obtained (a) by replacing all descriptive constants of A uniformly by

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[16] For further details cf. Coffa (1991) and Creath (1998). The definitions were meant to be given for an object-language in a more powerful (syntactic) meta-language.

[17] Tarski (1935). It can be said that Carnap (1937) defined truth for formalized logico-mathematical languages, though, unlike Tarski, he did not realize that much the same procedure can be used to define truth also for languages containing descriptive terms – cf. Coffa (1991) and Creath (1998).

variables of appropriate types and (b) universally closing the matrix so obtained with respect to each free variable introduced.

The definitions of contradictoriness and logical consequence for such sentences would have to be modified accordingly. Here, again, Carnap's procedure is recognizably quantificational in its spirit. In the next section, however, we shall have an occasion to see that this definition might eventually require a non-substitutional rendering of universal quantifiers supposed to close the purely logical matrix.

3.2. The objection from persistence violation

To sum up what has been said so far: substitutional accounts hold that consequence obtains between a (possibly empty) premise-set and the conclusion if there is no counter-example having the same logical form, where logical form is determined by the fixed logical terms and the pattern of remaining non-logical elements that are treated schematically. There is, mind you, no trace of modality in the explanations. Indeed, the appeal of quantificational approaches lies in the fact that we seem to need only appropriate quantifiers plus the notion of plain truth in order to account for logical properties in terms that are not philosophically contentious. In addition, quantificational approaches appear to do justice to the powerful intuition that logical properties are formal in nature: *if they apply to something, they apply to anything of the same form*. Logical status is, in this specific sense, *exceptionless*.

That being said, substitutional strategy of the sort I have reviewed may intuitively overgenerate, as it hinges on the expressive capacity of the underlying language. As Etchemendy put it, it is a plausible condition on an adequate account of logical consequence (truth) that:

> The property of being logically true with respect to a given F [class of fixed logical terms] should persist through simple expansions of the language ... [and] the property of not being logically true should persist through contractions of the language. (Etchemendy 1990, 30)

The same holds, *mutatis mutandis,* for consequence. Reductive substitutional accounts of logical consequence appear to violate this adequacy condition, at least when they are framed in a linguistic framework.[18] Viewed

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[18] Unlike Bolzano's (1837/1972) original account, which was designed for non-linguistic propositions and their component ideas, whose number as well as identity is independent of common languages, hence not constrained by their expressive limits.

from one direction, some arguments (sentences) in some language L can turn out valid (logically true) on the linguistic substitutional account, only because L does not have enough expressions to express a genuine counter-example – available in simple expressive expansions of L. From the opposite direction: some arguments (sentences) in L that are invalid (logically false) on the substitutional account, given that L has enough expressions to express a genuine counter-example, can become valid in its contractions (or sub-languages) lacking expressive resources to frame a counter-example.

Essentially this kind of objection was put forward by Tarski (1936) in his classic article on logical consequence. He says there that an adequate account of this notion should capture the following property called "the condition F":

> X follows logically from K only if X' is true whenever every member of K' is true, where X', K' differ from X, K only by replacement of all constants except logical constants (cf. Tarski 1936, 415).

Tarski then rushes to point out that this is a sufficient condition of X following logically from K, only if the underlying language has sufficient resources to designate any (type-theoretic or set-theoretic) object its quantifiers range over. But he deems this idea patently absurd, if only because there are many more set-theoretic objects than there are linguistic expressions.

3.3. Tarski's interpretational account circa 1936

Tarski's remedy was not to reject the basic idea behind the substitutional account but to improve on it so as to overcome the problem of persistence-violation. Like Bolzano, Russell or Carnap, he was arguably a supporter of a demodalized quantificational account of logical properties in terms of form, truth and generality. The challenge he faced was to find its proper formulation. Having argued that the F-condition is necessary but not yet sufficient for logical consequence, Tarski proposed what is nominally a model-theoretic account of logical consequence:

> The sentence X follows logically from the sentences of the class K if and only if every model of the class K is also a model of the sentence X. (Tarski 1936, 417)

Familiar as this account sounds Tarskian models are non-standard by contemporary standards. First, the definition was designed for type-theoretic logical systems common in the 1930s. Second, the consequence relation is defined for fully interpreted sentences, but their models are defined by detour through sentential functions obtained by uniformly replacing all their non-logical constants (in all their occurrences) by variables of fitting logical types. Thus, according to Tarski, starting with the sentence X and the class of sentences K, what we obtain via such uniform replacements will be the sentential function X* and the class of sentential functions K*. Whatever sequence of set-theoretic objects (of types appropriate to free variables of X*) satisfies X* is then called a model (or realization) of the sentence X.[19] Much the same can be said, *mutatis mutandis*, of the relation between K and K*: an arbitrary sequence of objects of fitting logical types that satisfies each sentential function in the class K* is called a model of that class of sentences.

Tarski thereby arrived at a definition of consequence that is remarkably close to Carnap's account introduced in Section 2.4. And his procedure is clearly a variation on the simple quantificational theme, as it reduces logical consequence (and logical truth) to plain truth plus generality. The chief difference is the consistent switch from substitutions to semantic valuations: what we uniformly vary are not extra-logical expressions but their set-theoretic values.[20]

It is worthy of comment that Carnap's account of logical notions for Language II could address the objection from persistence violation, owing to the fact that he was prepared to switch from substitutions to semantic values if need arises. This applies to his generalized definition of logical truth for Language II covering sentences with descriptive terms, in case one cannot reduce their analyticity just to analyticity of their substitutional variants but needs to quantify over semantic values proper. The need for this move was clearly pointed out to Carnap by Gödel in their correspondence on Carnap's early attempt to define analyticity (circa 1932), which was based on the substitutional reading of quantifiers. Gödel showed Carnap that his attempt was marred by circularity and proposed to fix the problem by treating second-order variables as raging over any property whatever defined over the individual domain (whether or not it can be

[19] Tarski's account of models of sentential functions builds on his celebrated recursive definition of satisfaction of sentential functions by (infinite) sequences of appropriate objects, given in Tarski (1935).

[20] Also, unlike Carnap, Tarski does not presuppose the generic consequence-relation defined as a closure upon inference rules (including rules, such as the omega-rule, that are infinitary in character).

named). Carnap incorporated this into his amended account in the *Logical Syntax*, where he stipulated the range of second-order variables to be the power set of the countable set of numerical expressions of Language II.[21] Much like Tarski, Carnap could in this way transcend the problem of persistence violation.[22]

4. One world is not enough: the objection from overgeneration due the fixed domain

Russell, Carnap and arguably also Tarski in the 1930s all had the idea that quantifiers are logical terms that cannot be varied via substitutions or interpretations. This is built into their logic in that individual variables are determined to range over one fixed domain of all individuals.[23] In fact, they all used to work with type-theoretic deductive frameworks devised to keep the spirit of the logicist reconstruction of the classical mathematics while avoiding Russell-type paradoxes.

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[21] Thus note that Language II, compared to Language I, is a very powerful formalism capable to embody a substantive amount of classical mathematical reasoning. With it Carnap embraced the idea of higher-order logic as a framework for mathematics. Now, while in the first-order fragment of Language II there are enough names (numerals) for every object in the domain of first-order quantifiers, the substitutional strategy is out of place with regard to higher-order fragments of Language II, as there are not enough names for every object in the intended domain of higher-order quantifiers (say for all properties/sets defined over the set of all individuals). See the discussion in Awodey – Carus (2007).

[22] Viz. the richness of objects to be found in the transfinite type-theoretic hierarchy – of which Language II is a part – that could be used in possible semantic valuations.

[23] Over this domain, then, domains for higher-order variables are defined, if the language is, as was then usual, type-theoretic. Whether Tarski (1936) propounded a fixed-domain or variable-domain conception of models is a vexed question. The fact is Tarski did not say there that different interpretations of sentential functions can be based on different domains. The discussion of this historical issue would take me too far afield. Let me just say that I think that Etchemendy (1990) is right that Tarski's account does not involve variable domains of interpretations. Consult Mancosu (2006; 2010), Bays (2002) or Corcoran – Sagüillo (2011) for more detailed arguments in favour of the view that Tarski held a sort of fixed-domain conception of models throughout 1930s, which are considerably more sensitive to the subtleties of Tarski's position in the historical context. Bays (2002), Mancosu (2006; 2010) and Schiemer – Reck (2013) also show that Tarski had resources that allowed him to simulate effects of domain-variation so that he could frame close enough versions of important metatheorems (such as completeness, etc.).

The trouble with this view is that once the domain of quantification is fixed, logical properties would seem to depend on its size in a way that is intuitively problematic. A notorious example is the sentence $\exists x \exists y \neg(x = y)$, which is logically true, if the domain contains at least two things, but logically false, if it contains only one thing, because it does not contain any non-logical element to be varied. Accordingly, if the fixed domain contains two or more things, any inference that has this sentence as a conclusion is logically valid. Generally, an inference of the following type that contains only cardinality-sentences about the least number of existing things

$$\frac{\begin{array}{c}\exists x_1 \ldots \exists x_n[\neg(x_1 = x_2) \wedge \ldots \wedge \neg(x_1 = x_n) \wedge \neg(x_2 = x_3) \wedge \ldots \\ \wedge \neg(x_2 = x_n) \wedge \ldots \wedge \neg(x_{n-1} = x_n)]\end{array}}{\begin{array}{c}\exists x_1 \ldots \exists x_{n+1}[\neg(x_1 = x_2) \wedge \ldots \wedge \neg(x_1 = x_{n+1}) \wedge \neg(x_2 = x_3) \wedge \ldots \\ \wedge \neg(x_2 = x_{n+1}) \wedge \ldots \wedge \neg(x_n = x_{n+1})]\end{array}}$$

is logically valid or invalid depending on the size of the fixed domain. Thus, for $n = 2$ and the fixed domain containing just two objects, the inference is invalid, since, in that case, the premise is true and the conclusion false. But if $n = 2$ and the domain contains at least three things, the inference is valid, the premise and conclusion being both true.

What this consideration seems to show is that, on the assumption that the quantifier is a fixed term picking out the universe of all existing individuals (or a fixed portion of it), quantificational accounts make logical properties dependent on the extra-logical fact of how many things there are in the fixed domain.[24] That is to say, any true cardinality sentence is logically true, and any false cardinality sentence is logically false, since there is nothing to be varied. Yet cardinality sentences, even if non-contingently true or false, do not seem to be true or false on purely logical grounds.[25]

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[24] With the possible exception that the domain is to be non-empty, though even this is over the heads of proponents of free logics. Already Russell (1919, 203, n. 1) was uneasy about this specific assumption, which was derivable from the axioms of *Principia Mathematica*.

[25] Such systems were based on an infinite individual domain, but this could hardly be otherwise, if the aim was to reconstruct the body of classical mathematics in them. This old-fashioned project is deemed *passé* today. Indeed, one tends to think that the axiom of infinity compromised it from the start. That said, it is to be noted that this was not the received view in the late 1920s and early 1930s. Thus Carnap and Tarski in the 1930s – both with sympathies to the general logicist idea – were prepared to treat (at

This raises the following problem for all quantificational accounts discussed so far: it seems that logical properties should persist not just under contractions and expansions of the non-logical vocabulary, but also under possible contractions and expansions of the domain of quantification.

5. Quine's parsimonious approach: substitutional account vindicated?

Quine (1970/1986) is the last proponent of the quantificational approach whose views merit our attention. Interestingly enough, he had a sort of response to both aforementioned overgeneration objections, based on his version of the substitutional approach.[26] He restricted his substitutional account to languages regimented in the austere first-order idiom (lacking primitive identity-symbol, individual terms and function symbols), for which he distinguished five accounts of logical truth and consequence: in terms of *structure*, *substitutions*, *models*, *proof* and *grammar*. The first account states that A is a logically true sentence if it is true by virtue of its logical structure alone. Since A's structure is revealed via replacing its predicates by schematic letters, Quine says that we can just as well define logical truth and consequence in the second way:[27]

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least sometimes) issues of cardinality as logical issues. So the discussion in this section is to be read in this light. Note, however, that Russell himself (1919, 202–203) viewed the axiom of infinity as problematic (along with the infamous axiom of reducibility) on the ground that it seems to be an extra-logical postulate. In his classic review of logicism Carnap (1931) had voiced a similar worry with respect to the traditional logicist program; yet, somewhat later, in Carnap (1937) he did not seem to be worried about assuming the axiom of infinity (pursuing already his quasi-syntactic and pluralist approach to logic without any clear-cut boundary between logic and mathematics).

[26] See Quine (1970/1986, 53-56). Compare also Quine (1950/1982, Ch. XIII). Quine mentions that the completeness theorem shows that, in case of first-order languages, logical truth and consequence can be adequately (extensionally correctly) approached in terms of proof and provability. But he does not prefer the proof-theoretic approach, on the ground that, unlike the general substitutional approach, it is arbitrary to the extent it depends on the choice of this or that proof-system.

[27] See Quine (1970/1986, Ch. 4). Simple open sentences play in Quine's regimented language the role of terms in Bolzano's unregimented language. Further measures are to be taken to avoid possible collision of variables. For a discussion see McKeon (2004).

(i) A^* is a logically valid schema iff all its instances obtained via admissible substitutions – of (open) sentences for its simple sentential schemata – turn out to be true sentences.
(ii) A is a logically true sentence iff it is such an instance of some logically valid schema A^*.
(iii) B follows logically from A_1, ..., A_n iff "If A_1 and ... and A_n, then B" is a logically true sentence.

The third account, which is Quine's version of the model-theoretic account in terms of set-theoretic interpretations of non-logical elements over varying domains, goes roughly like this (cf. Quine 1970/1986, 51-52):

(i) Open sentence $S(A^*)$ is a set-theoretic analogue of the schema A^* iff $S(A^*)$ is obtained by uniformly replacing every simple sentential schema of the type $P(x_1,...,x_n)$ that occurs in A^* by a corresponding set-theoretic construction of the type $(x_1,...,x_n) \in \gamma$, γ being a variable ranging over sets.
(ii) Model-sequence $M = (D, \alpha, \beta,...)$ of (n-dimensional) sets defined over the domain-set D satisfies the set-theoretic analogue $S(A^*)$ iff $S(A^*)$ comes out true when D is assigned as the range of its individual variables and the sets α, β,... are assigned (in the requisite order) to the set-variables occurring in $S(A^*)$.
(iii) Model-sequence $M = (D, \alpha, \beta,...)$ satisfies the schema A^* iff M satisfies its set-theoretic analogue $S(A^*)$.
(iv) A^* is a logically valid schema iff every model-sequence M satisfies A^*.

The clauses for logical truth and consequence may then remain the same as in the previous account.

Once we have the two accounts in place, the objection from persistence-violation can be reformulated: due to a lack of expressive power on the part of the object-language it might happen that the first account would count some sentences as logically true (having no substitutional counter-examples to them) that have set-theoretic counter-models. This, Quine admits, holds for expressively impoverished regimented languages. However, he has an argument that in the case of a canonical first-order language expressively adequate to elementary arithmetic there is no shortage of substitutional counter-examples *vis-à-vis* set-theoretic counter-models. So his substitutional account of logical truth does not overgenerate with respect to the account of logical truth spelled out in terms of set-theoretic interpretations.

What he set out to show is that the following equivalence holds for an arbitrary first-order schema S:

> S is true under all admissible substitutions iff no admissible set-theoretic interpretation is a counter-model of S.

He notes that the right-to-left direction of the equivalence

> If no admissible set-theoretic interpretation is a counter-model of S, then S is true under all admissible substitutions

holds due to a version of Gödel's completeness theorem for first-order logic (see Gödel 1930). The theorem states that every S that is true under all admissible set-theoretic interpretations (so has no counter-model) can be derived in a visibly sound first-order proof-system such that every provable schema is assured to have only true substitution-instances. Quine's argument for the left-to-right direction

> If S is true under all admissible substitutions, then no admissible set-theoretic interpretation is a counter-model of S

draws on two crucial meta-theorems. Let us first reformulate this conditional equivalently as follows:

> If some admissible set-theoretic interpretation is a counter-model of S, then S is false under some admissible substitution.

According to the Löwenheim-Skolem fundamental theorem, if a first-order schema has a model (counter-model) at all, it has a countable model (counter-model) in the domain of positive integers. So what we have to substantiate is this:

> If S has a countable counter-model in the domain of positive integers, then S is false under some admissible substitution.

Quine points out that Hilbert and Bernays showed that countable models or counter-models can be expressed by elementary number-theoretic open-sentences (available, recall, in Quine's preferred language – see Hilbert – Bernays 1934). Such open-sentences are exactly what one would need to frame falsifying substitution-instances of S if S had a countable counter-model. So we have:

> If S has a countable counter-model in the domain of positive integers, then S is false under some admissible substitution of open-sentences of elementary number theory.

Put all this together by chaining of implications and you have proved the left-to-right direction of Quine's equivalence.[28]

Incidentally, Quine also had a response to the second overgeneration objection, though he did not formulate it explicitly himself.[29] The clue lies in the fact that he did not treat the identity sign "=" as a primitive logical symbol of his canonical language (as is usual in first-order languages with identity) but as a defined symbol that expresses *indiscernibility with respect to all n-adic predicates* of the object-language. Consequently, sentences containing only quantified individual variables, sentential connectives and '=' are purely logical only by appearance, that is, until we replace the defined identity sign '=' with its definiens involving descriptive *n*-adic predicates of the object-language.[30] Once so expanded, it can be shown that cardinality sentences like $\exists x \exists y \neg (x = y)$ admit of substitutional counter-examples.[31] In this way, Quine could sustain his contention that, for a sufficiently expressive first-order language at least, his parsimonious substitutional account is all one needs.

Quine's vindication of his substitutional account raises a couple of questions. First, one could worry that his proposal makes logical properties dependent on substantive matters alien to logic, for an elementary number theory is needed to provide an adequate theory of sentences (finite strings) and substitutions, which is in turn equivalent to the theory of finite sets. I doubt, though, that this worry would have bothered Quine. Granted, his explication of logical properties brings in its own ontological commitments. But this is as it should be, by Quine's lights, because any theoretical ac-

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28 However, Boolos (1975, 52-53) argues that Quine's argument is correct for logical truth but, for logical consequence, his proof of extensional adequacy goes through only if the-premise set of an argument is finite (or at least arithmetically definable).

29 Shapiro (2000, 339) suggests this Quinean way out and McKeon (2004) details it.

30 Quine (1970/1986, 63) calls this procedure "exhaustion of combinations".

31 That said, Quine also argues that the basic laws of reflexivity, transitivity and symmetry of identity are logical truths in his substitutional sense, even though "=" is a defined binary predicate. Cf. Quine (1970/1986, 63-64). McKeon (2004) discusses some technical problems with Quine's idiosyncratic treatment of identity. Perhaps the most important philosophical point that many commentators have mentioned is that Quine makes identity of individuals relative on a richness of languages, since what is indiscernible with respect to all non-logical *n*-place predicates of one language, may well be distinguishable with respect to all non-logical predicates of a richer language. And that sounds counter-intuitive: Quine's '=' does not express identity after all.

count (the model-theoretic included) is bound to make some ontological commitments. And Quine is quick to remind us that the ontological costs of his substitutional approach are modest compared to the rival model-theoretic account formulated in terms of varying set-theoretical valuations (including varying domains of such valuations):

> [...] it renders the notions of validity and logical truth independent of all but a modest bit of set theory; independent of higher flights. (Quine 1970/1986, 56)[32]

Second, and more importantly, Quine's vindication of the substitutional account relies on fundamental meta-results for first-order predicate logic and it does not therefore carry over to languages regimented in higher-order predicate calculi. This would not have bothered Quine, who contended – partly on independent grounds – that second-order predicate calculus is a set theory in sheep's clothing, and hence no pure logic (see Quine 1970/1986, 66). Still, if one does not feel comfortable with Quine's fairly restrictive view of the realm of pure logic, one would have to seek another idea.

6. Conclusion

Where does this leave us? In light of the discussion so far, two reasonable desiderata on plausible accounts of logical properties have emerged that can be spelled out as follows:

(1) logical properties of (sets of) sentences had better persist under subtractions and expansions of the non-logical vocabulary;
(2) they should also persist no matter what sequence of values of appropriate types we assign to their non-logical elements – whatever possible domain of application those values may come from.

[32] Note that Quine's strategy of assuring extensional adequacy of his substitutional account already assumes that the first-order object-language is rich enough to embed elementary number theory. A related complaint could then be that, unlike the model-theoretic account assuming domain-variation, Quine's account makes logical properties dependent on the immanent ontological assumptions of the object-language itself. Quine, I guess, could retort that even the model-theoretic rival account makes some (indeed, much heavier) assumptions, albeit at the meta-level.

Only in this way, one might argue, could we hope to do justice to the intuition that logical validities as well as truths are topic-neutral and hence should not depend on substantial assumptions, be they empirical or mathematical. The problem is that the quantificational strategies fail to meet one or both of those desiderata.

When we set aside the objection from overgeneration due to the persistence violation as something that interpretational accounts eventually allow us to overcome, the real source of trouble seems to lie in the fact that all quantificational accounts so far reviewed treat quantifiers in a misguided way: viz. as rigidly ranging over one fixed domain, irrespectively, as it were, of the context of application. This, then, prevents them from doing justice to the intuition that specifically logical consequences and truths – conceived of as formal and topic-neutral in nature – are sensitive to the logico-semantic profile of sentences but insensitive to the specific contents associated with their descriptive vocabulary (if any) or specific domains they may be applied to.

That we can do better than this is the idea driving the standard model-theoretic account of logical consequence and related notions, which explicitly allows domains to vary across admissible semantic interpretations of language. I would eventually argue that this is the most promising quantificational approach on the market, not least because it provides formally rigorous explications of logical properties – relative to a principled account of the semantic behaviour of certain traditionally distinguished logical constants – that make room for fruitful metatheoretical comparisons between the semantic and the deductive side of logic.

This is a delicate issue – and one that has provoked much controversy recently – that deserves a separate discussion, because a mutated version of Wittgenstein`s principal challenge to Russell`s account is in a way still with us. Thus, if logical relations and properties are to be construed as formal and topic-neutral, it would seem that they should not be contingent on substantial truths. Yet even the model-theoretic approach appears to make them contingent on substantive matters, this time in the form of specific background set-theoretic assumptions.

This and closely related issues – such as what, in general, we can reasonably expect from fruitful meta-theoretical explications of central logical notions such as consequence – will be addressed in the concluding part of my explorations of the quantificational tradition.

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