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Quantificational Accounts of Logical Consequence I: From Aristotle to Bolzano[1]

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ABSTRACT: So-called quantificational accounts explicate logical consequence or validity as truth-preservation in all cases, cases being construed as admissible substitutional variants or as admissible interpretations with respect to non-logical terms. In the present study, which is the first from three successive studies devoted to quantification accounts, I focus on the beginning of systematic theorizing of consequence in Aristotle's work, which contains the rudiments of both modal and formal accounts of consequence. I argue, *inter alia*, that there is no evidence for the claim that Aristotle propounded a quantificational account, and that for a full-fledged quantificational approach in a modern style we need to turn to Bolzano's substitutional approach, whose motivation, structure and problems are explained in the second part of this study.

KEYWORDS: Aristotle – Bolzano – form – logical consequence – modality – validity.

1. Introduction

Semantic approaches to consequence typically emphasize truth-preservation, at least as a minimal requirement. An influential semantic tradition, with a long history attached to it, is premised on the idea that a

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consequence relation of special concern for logic requires more than minimal truth-preservation: i.e. that it does not happen, as a matter of fact, that the premises *Ps* all hold and the conclusion *C* does not hold. Rather, what is required is that *C* be true *whenever*, that is, *in all cases* in which *Ps* are jointly true. Within this tradition, furthermore, consequence of special concern for logic is a formal matter in at least the following sense: if *C* follows from *Ps*, any equiform argument *Ps*/C** is such that *C** follows from *Ps**. Accordingly, *cases* are construed either as admissible substitutions for all non-logical elements occurring in *Ps/C* or as admissible semantic interpretations (valuations) of such elements. This, then, presupposes a division of elements making up arguments into the distinguished logical terms (those not subject to substitution or reinterpretation) and the non-logical terms (those subject to such variations).

On the face of them, both substitutional and interpretational variations on this basic "quantificational" theme highlight the formal aspect of logical consequence at the expense of the modal/epistemic aspect – viz. *necessitation* or *guarantee* of a sort – that often resurfaces in informal glosses on consequence.[2] Their proponents have usually deemed this a praiseworthy virtue, whereas the critics lamented that this reductionist manoeuvre prevents them from capturing the very essence of consequence. The controversy over quantificational approaches, their *pros* and *cons*, is very much alive these days, and I think it deserves careful philosophical scrutiny. In three successive studies, starting with the present one, I set out to reconstruct the milestones in the quantificational tradition, examining in a systematic manner seminal contributions of Aristotle, Bolzano, Russell, Tarski, Carnap, Quine and the standard model-theoretic approach dominant today. Having discussed their merits and demerits *vis-à-vis* a battery of objections, I shall eventually argue (a) that quantificational accounts ultimately depend for their plausibility on their account (if any) of logico-semantic structure; and (b) that, in this respect, the model-theoretic ac-

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[2] They are called *quantificational*, because they explicate properties and relations such as logical truth and consequence in terms of truth of a certain universal generalization over appropriately construed cases. As far as I know, the label "quantificational accounts" was first coined by Etchemendy (1990, 98), who formulated what is perhaps the most influential criticism of them. I shall deal with his objections in detail in the second and third study.

count is the most promising of them – not least because it offers formally rigorous explications of logical relations and properties, based on a principled account of semantic contributions of distinguished logical expressions, that make room for fruitful theoretical comparisons between the semantic and the deductive side of logic.

That, in a nutshell, is my overall agenda. In the present study, I start my investigations of the "quantificational tradition" by focusing on the very beginning of systematic theorizing of consequence in Aristotle's path-breaking work, which arguably contains the rudiments of both modal and formal accounts of consequence. It is pointed out, though, that we lack evidence that Aristotle embraced a quantificational version of the formal account. The situation, I submit, did not dramatically change with the Stoics, though anticipations of the quantificational theme can be found in the writings of ancient commentators and medieval logicians who distinguished formal from material consequences. Nevertheless, for a full-fledged quantificational approach we need to turn to Bolzano's substitutional account, whose rationale, structure and problems I explain in the second part. This marks a true beginning of the story I want to tell, as virtually all the central issues concerning the nature and adequacy of quantificational approaches – to be addressed in the second and third study – can be motivated with reference to Bolzano's approach.

2. Validity and consequence: modality and form

To trace back the characteristic elements of quantificational approaches, we are well advised to start right with Aristotle's pioneering contributions. In the classic passage from the *Prior Analytics* he explains *deduction* (*sullogismos*) as a valid (conclusive) argument of a sort, in which holding of premises is a sufficient guarantee of the truth of the conclusion:

> *Aristotle's account of validity* (AAV):
> [...] deduction is a speech in which, some things having been supposed, something other than what has been supposed results of necessity from their being so. I mean by 'from their being so' resulting through them, and by 'resulting through them,' needing no term from outside for the necessity to arise. (Aristotle 1964, 24b18-22).

Now the phrase “results of *necessity* from their being so” may be taken to indicate that AAV-valid are all those arguments in which the premises necessitate (so guarantee) the conclusion. Thus broadly interpreted, AAV would seem to coincide with the following account commonly mentioned with approval by philosophers and logicians:

> *Modal account of validity* (MAV): argument is valid iff it is impossible that all its premises are true and (at the same time) its conclusion is false.

The truth, however, is more complex than that, as many have noted. For MAV subsumes circular arguments (including trivialities of the form *p; therefore p*), one-premise arguments, as well as arguments whose premises might have no connection to conclusions (allowing *ex falso/contradictione sequitur quodlibet* as well as *verum ex qoudlibet sequitur*).[3] AAV-valid arguments, on the contrary, are non-circular and multi-premised. Also their premises have to be sort of relevant to their conclusions in that the latter must result *through* the former, that is, *in virtue of* the premises holding (ruling out both *ex falso/contradictione sequitur quodlibet* and *verum ex qoudlibet sequitur*).[4]

At any event, what is conspicuous by its absence from the explicit dictum of AAV – given that Aristotle is the founder of logic as a science of formally-logically valid reasoning – is some indication that necessary truth-preservation is connected to (even grounded in or guaranteed by) forms displayed by valid arguments. Whether implicitly intended by Aristotle or not, some such restriction of AVV is called for to distinguish those arguments whose validity is a specifically logical matter.

Maybe, then, we are well advised to read in between the lines, because in Aristotle’s theory of narrowly syllogistic deductions the formal aspect plays a rather prominent role.[5] In delimiting the classes of such valid argu-

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[3] Aristotle conceived of valid reasoning as a tool for potentially acquiring new knowledge (or at least a probable opinion), provided that one has knowledge of (reasons/evidence/grounds of a sort for) premises. Clearly, circular arguments are cognitively impotent in this respect, being question-begging.

[4] Cf. Smith (1994), Corcoran (2008), Woods – Irvine (2004).

[5] Narrowly syllogistic are of course arguments with two premises and one conclusion, each component belonging to exactly one of the four notorious categorical kinds: *Every X is Y*; *Some X is Y*; *No X is Y*; *Not every X is Y*.

ments – viz. syllogisms – Aristotle made a systematic use of schemata with the following property: *no particular argument that is an instance of such a schema has false conclusion when its premises are all true*. The fact that arguments instantiating such schemata (necessarily) preserve truth is transparently displayed in their forms, as represented by schemata – all topic-specific elements with no bearing on truth-preservation (the "matter" of argument) being treated schematically.

That said, there is no conclusive evidence for the claim that Aristotle was ready to get rid of the modal element by defining argument to be valid if it is an instance of a valid schema, while explaining *valid schema* in the following reductionist (non-modal) manner:

> Argument-schema is valid iff no admissible substitution-instance of it – obtained via uniform substitutions of all schematic letters (in all their occurrences) by descriptive elements of fitting types – has true premises and false conclusion.

So understood, valid schema is *exceptionless*, that is, without counter-examples. We shall have an occasion to see that this is a strategic manoeuvre *par excellence* of several quantificational accounts of validity and related logical traits. In fact, some prominent commentators took Aristotle to have approved of this manoeuvre. Thus, in his pioneering study of Aristotle's logic, Łukasiewicz contends that "The Aristotelian sign of necessity represents the universal quantifier..." (Łukasiewicz 1957, 11).[6] Nevertheless, the evidence adduced by him for this claim is less than conclusive. For all we know, Aristotle seemed to think that truth-preservation characteristic of valid arguments involves necessitation as an essential-irreducible aspect, the task of logic being to show that and how this necessitation is connected to the forms which valid arguments share with equiform arguments.

In this respect, the situation did not dramatically change with the Stoics who, following Chrysippus, distinguished syllogistically valid arguments

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[6] Cf. Bolzano (1972, § 155; 219-220). According to Łukasiewicz, Aristotle subscribed to this quantificational approach in the following version (here applied to Barbara-style consequence): *For all X, Y, Z, if X belongs to all Y, and Y belongs to all Z, then X belongs all to Z*. Łukasiewicz claims that Aristotle did not formulate syllogisms as inferences at all but as material implications (Łukasiewicz 1957, 2), which fact apparently fits his own interpretation.

(*syllogistikoi logoi*) as a proper subclass of the broader class of valid arguments (*perantikoi logoi*) explained as follows (cf. Sextus Empiricus *PH* II 113, 137, 138; Diogenes Laertius VII 77-78):

> *The Stoic account of validity* (SAV): *P_1…P_n; therefore C* is a valid argument iff its corresponding conditional proposition (assertible) *If P_1 & … & P_n, C* is true.

This account also involves an essential modal ingredient, as the truth-condition of the conditional was explained as consisting in *conflict – incompatibility*, hence *impossibility* of a sort – between the antecedent and the negation of the consequent.[7] Somewhat in Aristotle's style, syllogistically valid arguments, as a subclass of SAV-valid arguments, divided into "undemonstrable" (evident) and those reducible to them. A significant difference, though, was that undemonstrable syllogisms were of exclusively propositional type, all other syllogisms reducing to them by means of four specific inference rules (called *themata*). Importantly, when it came to delimit five classes of undemonstrable syllogisms, the Stoics had recourse to so-called *modes* representing their forms (basically, schemata with ordinal numbers standing in for whole propositions; *modus ponens* being one of them: *if the 1^{st}, the 2^{nd}; but the 1^{st}; therefore the 2^{nd}*).[8] This classificatory function of modes makes it plausible to suppose that the Stoics were well aware of the

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[7] The rationale for requiring such a relationship probably was that the Philonian truth-condition (excluding only the case of the antecedent being true and the consequent untrue) and the Diodorean truth-condition (it never happens that the antecedent is true and the consequent untrue) were deemed too weak, as both could be satisfied also when *C* has nothing to do with *P_1 & … & P_n* (when there is no connection between the premises and the conclusion of the original argument). Cf. Gould (1974, 157-162). But the Stoic conception of possibility (or impossibility) is oriented on what is physically (naturally) possible (impossible). And the emphasis on a "connection" between the premises (antecedent) and the conclusion (consequent) suggests a criterion of relevance that goes beyond mere impossibility of the former being true and the latter untrue (which could be the case with inconsistent premises or with a necessary conclusion).

[8] Note that their preferred style of delimiting five classes of indemonstrable arguments proceeded not via modes but rather via metalinguistic descriptions of the sort: *a first indemonstrable is an argument composed of a conditional and its antecedent as premises, having the consequent of the conditional as conclusion*. Latter Stoics distinguished up to seven types of indemonstrable arguments. Cf. Bobzien (1999, 127).

fact that all equiform syllogisms are valid.[9] Still, it is not clear that this should be read as a sign that they deemed them valid in virtue of form. Indeed, there is little evidence that the Stoics understood the relation of *something following from other thing(s)* as formally-based rather than ontologically-based (cf. Frede 1987, 103). As Michael Frede suggests, the prevailing intuition among the ancients seemed to be that it is an ontological relation between states of affairs – often insufficiently distinguished from the relation of the antecedent to the consequent of a true conditional. If so, the relation owes nothing to argument-forms – albeit it can be represented in arguments of certain forms (cf. Frede 1987, 103-104). Also, their assimilation of the consequence relation to the conditional is rather unhappy and makes for further troubles having to do with the vagueness of the very criterion of conflict, which does nothing to distinguish specifically formal from conceptual or empirical incompatibility.

Summing up, Aristotle and the Stoics realized that certain valid arguments are distinguished by the fact that they can be grouped together by means of schemata such that all particular instances of them are valid. But we lack support for the claim that forms, as represented by schemata, were understood as grounding validity of such arguments (and that the modal ingredient was to be reductively explained (away) in terms of the schemas' holding in all terms/cases/instances – or something of the sort).

3. Consequence and validity: analytical and logical

In the subsequent tradition it has become common to distinguish the narrower notion of *formal validity* from the broader notion of *material* or *analytical validity*. The latter is, roughly, necessary truth-preservation, but not in virtue of argument-form alone. On this view, all logically valid arguments are analytically valid, but not *vice versa*. The labels "material" and "analytical" have often been used in a similar way in the tradition with regard to "consequence" or "validity". The latter became more frequent only after Kant and Bolzano, while the former was common in the medieval tra-

9 Cf. Bobzien (1999, 130). She, though, seems to think that there is some basis for claiming that the Stoics thought that formally valid syllogisms are valid in virtue of their forms, the forms grounding their validity.

dition influenced by Aristotle. In the medieval tradition, specifically in the 14th century Paris school led by John Buridan, formal consequence was said to hold *in all terms*, that is, irrespective of how terms (forming the "matter") are varied;[10] whereas material consequence was said to hold in virtue of its specific "matter" (so not in all terms):

> A formal consequence is one that holds for all terms retaining the same form, or if you wish to speak carefully ... for which any equiform proposition which might be formed would be an acceptable consequence. For example, 'That which is A is B, so that which is B is A' ... A material consequence is where not every proposition of the same form is valid ..., e.g., 'A man runs, so an animal runs', because it is not valid with these terms: 'A horse walks, so wood walks'. (Hubien 1976, I.4)

For instance, in neither of the following arguments

(A)

$$\frac{\text{Bob is a dog}}{\text{Bob is an animal}}$$

(B)

$$\frac{\text{Every Dalmatian is a dog}}{\text{Every Dalmatian is an animal}}$$

can the conclusion be false when the premise is true. Yet, intuitively, this has not to do with their forms (marked by syncategorematic elements), but with the meaning-connection between the descriptive (categorematic) terms "dog" and "animal". That this specific meaning-connection underlies validity of A and B is revealed once we abstract out the specific contents of descriptive terms, replacing them with schematic letters in a uniform manner. We thereby obtain the schemata apparently admitting of many counterexamples:

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10 It could be said that Buridan with his allies came closest to the modern quantificational account of consequence in the substitutional style. However, even in their case, it does not seem likely that they (a) treated the property of *holding in all terms* as grounding validity (rather than as a test or criterion of it) or (b) that they wanted to reductively explain (away) the modal element (of necessitation). Cf. Read (2012).

x is X	Every Z is X
———	———
x is Y	Every Z is Y

It is sometimes claimed that material (analytical) validity reduces to logical validity, with some suppressed (and already implicit) premise(s) being inserted.[11] Compare again Buridan, who is quite explicit on the matter:

> No material consequence is evident except by reduction to a formal consequence by the addition of some necessary proposition. (Hubien 1976, I.4)

The idea is, presumably, that without tacitly presupposing some bridge-premise, the original premise-set simply does not provide any conclusive ground(s) for the conclusion. Applied to the case under consideration, it may be taken to suggest that inferences such as A or B are enthymematic, because needing the universal premise "Every dog is an animal" to count as *bona fide* conclusive arguments displaying the exceptionless forms:

x is X	Every Z is X
Every X is Y	Every X is Y
———	———
x is Y	Every Z is Y

It indeed seems that a prevailing tendency in the tradition – detectable in the work of Aristotle, the Stoics or medieval logicians – was to think, with Buridan, that validity of material consequences can be rendered evident only by reducing them to formal consequences that are complete. In general, however, it seems a dubious strategy to question the fact that the premises in A and B are relevant to their conclusions: for whoever understands and accepts the premise has thereby an excellent reason to accept their conclusions. Granted, then, this is so due to the occurrence of certain descriptive terms whose meanings are closely linked. However, it does not follow from this that any new premise is called for to guarantee that the necessary truth-preservation really arises in A or B. Note that, strictly speaking, even the inference

[11] Cf. Copi's (1968) classic logic textbook, in which he propounds the missing premise strategy.

(B)

$$\frac{\text{Someone is mortal}}{\text{Not everyone is not mortal}}$$

hinges on the meaning-connection between "someone" and "not everyone ...not". Should we say, by parity of reasoning, that this inference depends for its validity on a certain fact "from outside", perhaps the premise "For every X, if someone is X, then not everyone is not X?" There is a serious obstacle to this line of reasoning, namely a version of the argument of Lewis Carroll showing that this would start an infinite regress of adding yet further and further bridge-premises without end.[12]

The very thesis of the alleged dependence of analytical on logical validity is thus far from unproblematic. Even quite apart from that, one has to take into account the fact that analytical validities such as A or B – or even some factually-based inferences, as we shall see in due course – can, in a way, be construed as "formally valid". For we may represent their forms respectively by the schemata

$$\frac{\text{x is a dog}}{\text{x is an animal}} \qquad \frac{\text{Every X is a dog}}{\text{Every X is an animal}}$$

no admissible substitution-instance of which is a counter-example.

To be sure, one may insist that such inferences are "semi-formal" at best, on the ground that their validity depends on the fixed "topic-specific" terms and their connected meanings. For logic, it may be contended, is topic-neutral and in this sense also general. Or one may prefer to say that their validity depends on identities of entities – be they particular or general – referred to by topic-specific elements. This could have been Aristotle's way of looking at the problem under consideration. Thus that the concept/class *dog* is (in some mereological sense, say) contained in the concept/class *animal* has to do with the nature/essence of dogs, but nothing to do with logic proper, which is concerned solely with the most general structural features of reality. It is their dependence on such extra-logical – if perhaps metaphysically necessary – facts that makes A and B non-logical

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12 See Caroll (1895). A thoughtful discussion of this and related issues is Smiley (1994). Read (2002) argues against treating such arguments as enthymematic.

in character. Or so Aristotle seemed to think. On the other hand, a genuine logical feature could be, for instance, the transitivity of concept inclusion, which can be taken to underlie the validity of Barbara-style inferences: *X belongs to all Y; Y belongs to all Z; so X belongs to all Z.* Apparently, the actual identities and relations of the concepts *X*, *Y* and *Z* are irrelevant. What matters are the structural properties of *concept* (or *class*) *containment*, which are indicated by (or reflected in) logical particles (the form). Incidentally, this could also explain why Aristotle required premises of logically valid arguments to be connected with conclusions via certain shared elements (typically terms) so that both the relevance and necessary truth-preservation would be indicated solely by a certain structural pattern and not by meaning-linkages between topic-specific elements.

A *prima facie* challenge for this traditional way of distinguishing materially or analytically valid from logically valid arguments is to come up with a plausible demarcation of logical terms (notions) and their function or else to treat the division of terms into logical and non-logical as a non-absolute (and to some extent arbitrary) matter. Bernard Bolzano was well ahead of his times, being fully aware of this situation. He was for the latter option.

4. Bolzano's pure quantificational theme: modality explained away

Aristotle, the Stoics and medieval logicians were not bothered with consequence involving an irreducible modal element of necessitation. But, given that modalities of various kinds are perennially problematic notions, is it not desirable to free logic from such contentious elements by explaining them away or somehow reducing them to less contentious notions? A philosopher sympathising with this methodological strategy prefer to return to the vital idea that validity of an argument consists in the fact that its conclusion is true *whenever* its premises are jointly true. Then, of course, one has to spell out this informal *whenever-connection* without invoking modalities in any explicit or implicit manner.[13] Fortunately, one can have

[13] Be it in the informal manner of Aristotle's *locus classicus*, or in the modern *façon de parler of* possible-worlds. See Read (2002) who argues for an irreducibly modal account of consequence (validity).

recourse to the already mentioned generalization about the form of argument as represented by the relevant schema:

> *Substitutional account of consequence* (SAC): argument is valid iff it is an admissible substitution-instance of an argument-schema such that no argument instantiating the schema has true premises and a false conclusion.

Bolzano might have been the first to elaborate rigorously on this very idea in his account of logical validity and deducibility. The following passage deserves a full quote:

> Among the definitions of [the concept of deducibility] ... one of the best is that of Aristotle: 'a syllogism is a discourse in which, certain things being stated, something other than what is stated follows of necessity from their being so.' Since there can be no doubt that Aristotle assumed that the relation of deducibility can hold between false propositions, the 'follows of necessity' can hardly be interpreted in any other way than this: that the conclusion becomes true whenever the premises are true. Now it is obvious that we cannot say of one and the same class of propositions that one of them becomes true whenever the others are true, unless we envisage some of their parts as variable. For propositions none of whose parts change are not sometimes true and sometimes false; they are always one or the other. Hence when it was said of certain propositions that one of them becomes true as soon as the others do, the actual reference was not to these propositions themselves, but to a relation which holds between the infinitely many propositions which can be generated from them, if certain of their ideas are replaced by arbitrarily chosen other ideas. (Bolzano 1972, § 155, 219-220)

Here he explains the informal *whenever*-connection as preservation of truth under all admissible variations of the premises $A_1,\ldots,A_n$ and the conclusion B. In fact, he subscribed to the following direct version of SAC:

> Argument is valid iff no admissible variant of it (with respect to some variable element(s)) has true premises and a false conclusion.

Bolzano did not intend his notions of compatibility, validity and deducibility to apply primarily to sentences of a colloquial language, but to mind- and language-independent propositions (*Sätze and sich*).[14] More precisely, those notions had to take into account *kinds* or *classes of propositions* of the same *form*. For, unlike sentential schemata, propositions do not literally contain any undetermined elements indicated by variables of a sort. Consequently, the talk about variable component representations (*Vorstellungen an sich*) in the proposition P is to be understood as being about classes of P's variants – classes of propositions that differ from P at most in that they have different component representations in those places where P has component representations taken as variable, while sharing with P the structure/form of fixed elements.

Still, in order to single out propositional forms determining various kinds of equiform propositions (relative to a non-empty class of variable ideas), Bolzano often employed sentential schemata, speaking metaphorically of a variable element in a sentential-schema being replaceable by a representation of a fitting type.[15] The natural counterpart of his strategy at the linguistic level would thus be to focus on sentential-forms. In what follows I shall explain Bolzanian notions in this linguistic manner to keep it close to modern substitutional accounts as formulated by Carnap (1937/2002), Quine (1986) or Tarski (1936).

On Bolzano-style substitutional account the *whenever*-connection amounts to the preservation of truth under all admissible variations of the premises $A_1,\ldots,A_n$, and the conclusion B. Such an admissible variation is obtained via a substitution operation s on the set $\{A_1,\ldots,A_n, B\}$ satisfying the following conditions:

a) s operates on a non-empty set $V = \{a_1,\ldots,a_k\}$ $(1 \leq k)$ of elements occurring in the set $\{A_1,\ldots,A_n, B\}$ which are considered as variables;
b) s replaces every element $a_i \in V$ $(1 \leq i \leq k)$ by an element of an appropriate logico-semantical type (*salva congruitate*);
c) s is uniform: repeating occurrences of the same variable element a_i in $\{A_1,\ldots,A_n, B\}$ are everywhere replaced by the same element.

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14 *Propositions* (*Sätze an sich*) are objectively if abstractly existing truth-evaluable entities. For more details see Rusnock – Burke (2011), Siebel (2002) or LaPointe (2011).

15 For more background consult LaPointe (2011), Rusnock – George (2004) or Vlasáková (2005).

Notational convention: $s(A)$ is the result of applying s to A.[16]

On this basis we can define the relative notion of *deducibility* (*Ableitbarkeit*) corresponding, in spite of its name, to the broader semantic notion of *analytical consequence*, as well as the relative notion *logical deducibility* that is counterpart of the narrower notion of *logical consequence*. Here is my reconstruction of the first notion:[17]

> B is deducible from $A_1,\ldots,A_n$ w.r.t. V iff no admissible substitution s w.r.t. V is such that $s(A_1)$, ... , $s(A_n)$ are all true and $s(B)$ is false.

This, again, is a fairly broad notion that subsumes also our "analytically based" valid argument

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16 Bolzano imposed a few other constraints on admissible substitutions: (1) variable elements $a_1,\ldots,a_k$ in $\{A_1,\ldots,A_n, B\}$ must be simple (not considered decomposable into simpler components); (2) none of them is to be logical in character; and (3) if a_i is a subject-term of a sentence, then s has to replace it with a term belonging to the same semantical category that has a reference (or objectuality, be it concrete or abstract). This last restriction hangs in closely with two specific elements of Bolzano's theory of propositions. First, he thought that every proposition can be put into the tight subject-predicate form of the type "x has (an) X"; second, he did not allow for truth-value gaps holding that any proposition with "x" empty (non-objectual) is outright false. Given this semantic assumption, propositions of the form "x has X or x has not X" would not be analytically (or indeed, logically) true, unless we restrict the substitution-class for "x" so as to exclude empty terms. Morscher (2012) (viz. the technical appendix) has useful reconstructions of basic definitions closer to Bolzano's original approach. Rusnock – Burke (2011) point out that Bolzano allowed for the possibility of taking only certain occurrences of a term to be taken as variable, and note in this respect the analogy with Frege's substitutional strategy of obtaining predicates from sentences. I do not follow Bolzano in this.

17 In addition, Bolzano requires premises $A_1,\ldots,A_n$ to be *compatible* (*verträglich*) with B w.r.t. V, which notion he defines in terms of there being at least one admissible substitution s w.r.t. V such that $s(A_1)$, ... , $s(A_n)$ and $s(B)$ are all true. I leave out this significant element of Bolzano's original approach to keep it closer to the classic logic common today, according to which *ex falso/contradictione quodlibet* holds: anything follows from incompatible premises. Owing to this specific feature, Bolzano's account of consequence differs from modern logic in that it is obviously non-monotonic (addition of new premises to the premise-set might make the enlarged premise-set incompatible with the conclusion) and non-contrapositive (if premises are compatible with conclusion, but the negations of premises are not compatible with negated conclusion). *Reductio ad absurdum* arguments seem to be also out of place. For a discussion see Siebel (2002).

$$\frac{\text{Fido is a dog}}{\text{Fido is an animal}}$$

since, permuting only with respect to "Fido", we will never have true premise and false conclusion, under any admissible substitution with respect to this element. To put it slightly differently, this argument has no counter-example in the set of all admissible variants relative to "Fido" considered as its only variable element. Alternatively, there is no counter-example to the argument-schema (consisting of two sentential schemata):

$$\frac{\text{x is a dog}}{\text{x is an animal}}$$

It could occur to one that we might even say that the original argument is logically valid – its conclusion following logically from its premise – as it apparently displays the form represented by the above schema that has no admissible instance combining true premise with false conclusion. However, I have already emphasized that validity of this argument-schema differs from that of syllogistic schemata, in that it owes something to the meanings of descriptive, topic-specific elements ("dog" and "animal"). It is thus not valid solely on the basis of its topic-neutral logical skeleton

$$\frac{\text{x is (a) X}}{\text{x is (a) Y,}}$$

which has substitutional counter-examples.

Bolzano was well aware of this:

> [There are] propositions that are deducible (*ableitbar*) from other proposition by virtue of their sole form (that is, that are deducible insofar as we consider all the parts that do not belong to their form as variable). (Bolzano 1972, § 29, 141)

Accordingly, a narrower notion of logical consequence (deducibility) can be defined as follows:

> *B* is logically deducible from $A_1,\ldots,A_n$ iff *V* contains all and only the non-logical elements occurring in $\{A_1,\ldots,A_n, B\}$ and no admissible sub-

stitution s w.r.t. V is such that $s(A_1),\ldots,s(A_n)$ are all true and $s(B)$ is false.[18]

Our analytically-based argument does not meet the condition, since by varying any save the logical "is"-element we obtain many counter-examples, e.g.:

$$\frac{\text{Obama is a man}}{\text{Obama is a cosmonaut}}$$

Bolzano officially saved the detour through schemata – sentential-schemata being taken as no more than a useful heuristics – but his account of consequence suggests that an argument is logically valid if it is an instance of *at least one* purely logical schema that is exceptionless: without substitutional counter-examples. Similarly, an argument is (analytically) valid if it is an instance of *at least one,* possibly not purely logical schema having no admissible substitutional counter-example.

Once we have so explained logical consequence, it is possible to define logical truth as a limit-case: A is logically true iff A is deducible from the empty set of premises (or from any set of premises whatsoever). Bolzano, however, approached the matter in the reversed order. He first introduced the notions of (universal) *validity* and (universal) *contravalidity* for sentences, and only then defined analytical and logical consequence. On our preferred linguistic version of it, *validity* would be defined as relative to a non-empty set V of elements of A considered as variable:

A is valid w.r.t. V iff $s(A)$ is true, for every admissible substitution s w.r.t. V.

Given the same assumptions, relative *contravalidity* can be defined this way:

A is contravalid w.r.t. V iff $s(A)$ is false, for every admissible substitution s w.r.t. V.

Then relative *analyticity* resp. relative *syntheticity* can be defined as follows:

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[18] Bolzano again demanded that premises of a (logically) valid argument be compatible with its conclusion.

A is analytic w.r.t. *V* iff either *A* is valid w.r.t. *V* or *A* is contravalid w.r.t. *V*.
A is synthetic w.r.t. *V* iff *A* is neither valid nor contravalid w.r.t. *V*.

As a special case of analyticity we finally have:

A is logically analytic w.r.t. *V* iff all but the logical elements of *A* belong to *V* and *A* is analytic w.r.t. *V*.

Note that relative analyticity includes both relative analytical truth and relative analytical falsity. *A* can be said to be analytic *simpliciter* if there is at least one non-logical element of it taken as variable such that *A* is analytic relative to that element. In the same spirit, *A* can be said to be synthetic *simpliciter* if there is at least one non-logical element of it taken as variable such that *A* is synthetic relative to that element. Thus, for instance, the sentence

No female who is a philosopher is a bachelor,

is analytic *simpliciter*, since "philosopher" is a variable element of it such that no admissible replacement of it yields a false variant.[19] But it is not logico-analytic *simpliciter*: all its logical elements being fixed, there are permutations with respect to other variable elements of it – say "bachelor" – which result in false as well as true sentences:

No female who is a philosopher is a mother.
No female who is a philosopher is a man.

On the other hand, the sentence

Oscar is a philosopher or Oscar is not a philosopher

is logically (so analytically) valid, since it is valid with respect to *all* (hence *some*) of its variable elements, its logical skeleton (form) *x is X or x is not X* being fixed.

It could seem that Bolzano more or less develops, albeit in the specific and rigorous way, the ideas that were already familiar in the Kantian tradi-

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[19] Bolzano would add that admissible substitutions have to be such that the subject-term of a variant stands for something that exists.

tion. As regards logico-analytical propositions and their relation to broadly analytical propositions he said that:

> [...] no other than logical knowledge is necessary, since the concepts which form the invariable part of these propositions all belong to logic. On the other hand, for the appraisal of the truth and falsity of propositions like those given first [i.e. analytical – my insertion] ... a wholly different kind of knowledge is required, since concepts alien to logic intrude. (Bolzano 1972, § 148, 198).

At first blush, this looks traditional: logico-analytical propositions depend for their truth-value solely on logical concepts while broadly analytical propositions depend for their truth-value (also) on concepts alien to logic. The only notable difference would seem to be that Kant would say that logical propositions are purely formal and contentless in that their truth depends solely on a pure *a priori* form imposed by the mind.

On closer inspection, there are further significant differences. First, Bolzano's account is relative to a division of elements into fixed-logical and variable-non-logical. Now he surely had his own preferred list of logical elements (ideas),[20] but he was quite explicit that there may be no definite division (demarcation criterion) after all:

> This distinction, I admit, is rather unstable, as the whole domain of concepts belonging to logic is not circumscribed to the extent that controversies could not arise at times. (Bolzano 1972, § 148, 198-199).

Second, Bolzano was suspicious of the notion of a pure *a priori* form as it was used in the Kantian tradition, especially in service of drawing clear-cut boundaries between

(1) *analytic* judgements whose truth depends on the non-logical concepts involved in them (having *a priori* status because purely concep-

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[20] Bolzano's favorite list of logical items does no coincide with what logicians have usually in mind today. For one thing, recall that Bolzano thought that the standard logical form of any statement/proposition is the subject-predicative "x has (a) X". For him, "has" is a logical word that performs the role of copula connecting subject-term with the predicate-term. Or, for instance, the sentence "There are beautiful things" would be rendered as "The idea of beauty has objectuality", where the only non-logical term is "beauty", but "idea" and "objectuality" are logical notions.

tual – but not purely formal-logical – grounds suffice for their justification);

(2) *logical* judgements whose truth depends solely on the pure forms of (having *a priori* status because purely formal-logical grounds suffice for their justification);

(3) *synthetic* judgements whose truth depends on non-conceptual matters (having (a) *a posteriori* status if their justification requires also empirical grounds, or (b) *a priori* status if their justification requires pure intuition).

Note first that Bolzano thought that a proposition is analytical to the extent it *does not depend* for its truth-value on certain concepts (those variable *salva* truth-value). A logico-analytic proposition is a limit-case, because it does not depend for its truth-value on none except the logical concepts (all other elements being variable *salva* truth-value). In this specific sense, analytical and logico-analytical propositions are both "formal", the distinction between them depending on the division between logical and non-logical elements that is itself "rather unstable". Furthermore, even among true propositions that are synthetic by the Kantian taxonomy there are some, such as the following,

Every human who is male lives less than 200 years

that can be thought of as depending for its truth-value on a form of a sort. Thus the form

Every human who is X lives less than 200 years

happens to have only true instances, and, accordingly, the original proposition is true independently of the concept signified by "male". So Bolzano would classify it as analytic *simpliciter*, though it is obviously contingent and Kantians would no doubt say that it potentially expands our knowledge. So his conception differs significantly from the Kantian view.

All in all, from Bolzano's perspective there is no clear-cut distinction between the synthetic and the analytic on the one hand, and the analytic and the logico-analytic on the other. In his opinion, versatility and relativity pertaining to the very notion of *form* made it quite inapt to fulfil the high traditional expectations. Interestingly enough, he drew from this also epistemological ramifications, since he questioned the view according to which the truth of analytical and logico-analytical propositions cannot but be known *a priori* (either via purely conceptual knowledge or via logical knowledge of pure forms). For instance, though the ultimate objective ground for logical propositions (relative, that is, to the selection of logical concepts) lies in knowledge of purely logical objects, nevertheless, many logical laws (truths or inferences such as syllogistic figures) are actually accepted on empirical (inductive) grounds.[21]

5. By way of conclusion

Having said that Bolzano developed the first full-blooded quantificational account of logical properties in general, and consequence in particular, we should also note its ramifications and problems.

First, there is the issue of dependence of logical properties on the demarcation of distinguished logical elements. Bolzano's approach, it was said, is quite liberal, to the annoyance of those who would like to see application of logical properties as an absolute matter. On the other hand, the subsequent history of the subject has to some extent confirmed Bolzano's suspicion that "the whole domain of concepts belonging to logic is not circumscribed to the extent that controversies could not arise at times." The charge of relativizing consequence and related logical properties to a (somewhat arbitrary) selection of logical elements is thus hardly decisive, at least until someone convinces us that there is a principled demarcation criterion.

Second problem is that substitutional approaches in Bolzano's style render logical consequence, validity and related properties dependent on the actual richness of vocabulary. Clearly the range of equiform variants to a given argument varies with the range of possible linguistic substituends for its variable (non-logical) elements. Accordingly, it could happen that, if the vocabulary is sufficiently contracted, the argument might have only truth-preserving variants, whereas appropriately expanded vocabulary would generate an equiform counterexample to it. Intuitively, though, the argument

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[21] See Coffa (1991) for a critical discussion, and Rusnock – Burke (2011) for a corrective to Coffa's claim that Bolzano thought that the grounds for logical propositions come from empirical (inductive) evidence.

should remain valid (or invalid) no matter what expressions we add to or subtract from the vocabulary. It is well-known that Bolzano's original account in terms of propositions and ideas (*an sich*) does not face this problem, because ideas, as possible variands and substituends, are not limited in the way expressions are. But the problem is pressing enough for substitutional approaches in general (viz. cases of a non-denumerable infinity of objects not coverable by denumerably many terms).

Third, there is an obvious problem with sentences (propositions) in which there are no non-logical elements, so nothing to vary. Such sentences, if true, would have to be logically analytic. Yet there appear to be sentences couched in purely logical terms – perhaps "There is something" – whose truth does not seem to be grounded in logic but in contingent facts. This applies, *mutatis mutandis*, to arguments composed of sentences articulated in logical terms only: such an argument would have to be logically valid if either its premises are not all true or the conclusion is true (in the sense of material truth-preservation).

Closely connected to this is the last and arguably the most important charge to the effect that Bolzano's account cannot distinguish generalizations that are true as a matter of fact from generalizations that are necessarily true (analytically or formally), thereby providing for a deeply confused assimilation of the logical to the empirical. Though not everybody would be paralyzed by the suggestion that there may be no clear-cut boundary between the empirical and the logical – especially when the boundary is drawn by means of the traditional dichotomies between contingent and necessary or factual and non-factual – this charge would be deemed serious enough by many logicians and philosophers.

Bolzano's substitutional approach is not the only quantificational approach facing those problems. But this is another story, which I prefer to reserve for the next occasion.

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